Polygon Dissections and Standard Young Tableaux

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ABSTRACT

A simple bijection is given between dissections of a convex \((n+2)\)-gon with \(d\) diagonals not intersecting in their interiors and standard Young tableaux of shape \((d + 1, d + 1, 1^{n-1-d})\).

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For $0 \leq d \leq n-1$, let $f(n,d)$ be the number of ways to draw $d$ diagonals in a convex $(n+2)$-gon, such that no two diagonals intersect in their interior. For instance, $f(n,n-1)$ is just the Catalan number $C_n = \frac{1}{n+1} \binom{2n}{n}$. A result going back to Kirkman [3], Prouhet [4], and Cayley [1] (with Cayley giving the first complete proof) asserts that

$$f(n,d) = \frac{1}{n+d+2} \binom{n+d+2}{d+1} \binom{n-1}{d}. \quad (1)$$

K. O'Hara and A. Zelevinsky observed (unpublished) that the right-hand side of (1) is just the number of standard Young tableaux (as defined, e.g., in [5, p. 66]) of shape $(d+1, d+1, 1^{n-1-d})$, where $1^{n-1-d}$ denotes a sequence of $n-1-d$ 1's. It is natural to ask for a bijection between the polygon dissections and the standard Young tableaux. If one is willing to accept the formula for the number of standard Young tableaux of a fixed shape (either in the original form due to MacMahon or the hook-length formula of Frame-Robinson-Thrall), then one obtains a simple proof of equation (1). In this note we give a simple bijection of the desired type.

First we recall that there is a well-known bijection [2] between dissections $D$ of an $(n+2)$-gon with $d$ diagonals and integer sequences $\psi(D) = (a_1, a_2, \ldots, a_{n+d+1})$ such that (a) either $a_i = -1$ or $a_i \geq 1$, (b) exactly $n$ terms are equal to $-1$, (c) $a_1 + a_2 + \cdots + a_i \geq 0$ for all $i$, and (d) $a_1 + a_2 + \cdots + a_{n+d+1} = 0$. This bijection may be defined recursively as follows. Fix an edge $e$ of the dissected polygon $D$. When we remove $e$ from $D$, we obtain a sequence of dissected polygons $D_1, D_2, \ldots, D_k$ (where $k+1$ is the number of sides of the region of $D$ to which $e$ belongs), arranged in clockwise order, with $D_i$ and $D_{i+1}$ intersecting at a single vertex. If $D_i$ consists of a single edge, then define $\psi(D_i) = -1$, and set recursively $\psi(D) = (k-1, \psi(D_1)^*, \psi(D_2)^*, \ldots, \psi(D_{k-1})^*, \psi(D_k))$, where $\psi(D_i)^*$ denotes $\psi(D_i)$ with a $-1$ appended at the end.

Given a sequence $(a_1, a_2, \ldots, a_{n+d+1})$ as above, define a standard Young tableau $T$ of shape $(d+1, d+1, 1^{n-1-d})$ as follows. We insert the elements $1, 2, \ldots, n+d+1$ successively into $T$. Once an element is inserted, it remains in place. (There is no "bumpling" as in the Robinson-Schensted correspondence.) Suppose that the positive $a_i$'s are given by $b_1, b_2, \ldots, b_{k+1}$, in that order. The insertion is then defined by the following three rules:

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• If \( a_i > 0 \), then insert \( i \) at the end of the first row. (We write our tableaux in “English” style, so the longest row is at the top.)

• If \( a_i = -1 \) and the number of \(-1\)’s preceding \( a_i \) is given by \( b_1 + b_2 + \cdots + b_j \) for some \( j \geq 0 \), then insert \( i \) at the end of the second row.

• If \( a_i = -1 \) and the number of \(-1\)’s preceding \( a_i \) in not of the form \( b_1 + b_2 + \cdots + b_j \), then insert \( i \) at the bottom of the first column.

It is an easy exercise to check that the above procedure yields the desired bijection.

**Example.** Let the sequence corresponding to a dissection \( D \) (with \( n = 14 \), \( d = 6 \)) be given by

\[(4, 2, -1, 1, -1, -1, 3, -1, -1, 1, -1, -1, -1, -1, -1, -1, 2, -1, -1).\]

We have \((b_1, \ldots, b_7) = (4, 2, 1, 3, 1, 1, 2).\) We have printed in boldface those \(-1\)’s that are preceded by \( b_1 + \cdots + b_j \) \(-1\)’s for some \( j \). The corresponding standard tableau \( \psi(D) \) is given by

\[
\begin{array}{cccccccc}
1 & 2 & 4 & 7 & 10 & 11 & 18 \\
3 & 9 & 13 & 14 & 17 & 19 & 20 \\
5 \\
6 \\
8 \\
12 \\
15 \\
16 \\
21 \\
\end{array}
\]

**References**


