Homework 1 - due January 12th

1. It is 2027 and you are a professor at the Institute for Writing And Research Mathematics (WARM). After class at WARM one day, one of your students asks you to look over their “proof” for the problem

You have a white box with 60 white balls and a black box with 60 black balls. You take 20 balls from the white box, put them in the black box, mix the balls around and then randomly take out 20 balls and put them in the white box. In the end, which is larger: the number of black balls in the white box, or the number of white balls in the black box?

Proof. The numbers are the same because however many black balls move to the white box, that’s how many white balls move to the black box. For example, say the batch of 20 balls we move back to the white box has 11 black balls and 9 white balls, then 11 white balls stayed in the black box, by induction.

(a) In which of the following ways is this not a proof? Explain your answer.

- It is too vague to be a rigorous proof.
- It uses irrelevant concepts.
- It restates the claim we are trying to prove using it as an explanation.
- It is an example.

(b) You have a white box with 60 white balls and a black box with 60 black balls. You take 20 balls from the white box, put them in the black box, mix the balls around and then randomly take out 20 balls and put them in the white box. Then you do this again. In the end, which is larger: the number of black balls in the white box, or the number of white balls in the black box? Prove your answer.

Homework 2 - due January 19th

2. (a) One day in the coffee room, one of your fellow professors at WARM comes up to you and says “All babies have the same colour eyes! I have a proof!”

What is wrong with the following proof that they give you?

Proof. We are going to do a strong induction on the number of babies.

Base case: \( n = 1 \) is clearly true.

Induction step: Assume that every set of up to \( n \) babies has the same colour eyes. Consider a set of \( n + 1 \) babies sitting in a straight line. By induction the \( n \) leftmost babies have the same colour eyes. Similarly by induction the \( n \) rightmost babies have the same colour eyes. Then all \( n + 1 \) babies have the same colour eyes, as the rightmost baby and the leftmost baby have the same colour eyes as all the babies in between.

Hence, by induction, every set of \( n \) babies has the same colour eyes for all \( n \geq 1 \). Since the total number of babies is some number \( N \geq 1 \) the result follows.
(b) Prove that for every positive integer $n$,

$$1^3 + 2^3 + \cdots + n^3 = \left( \frac{n(n+1)}{2} \right)^2.$$