Maximum flows, minimum cuts, and longest paths

Steph van Willigenburg
Math 442-201 2019WT2

2 April 2020
**Networks**

**Definition**

Let $D$ be a connected digraph. Then it is a **network** if each arc $a \in A(D)$ has a weight, called **capacity** $c(a)$.

Let $v$ be a vertex. The sum of capacities of arcs coming in is the **in-flow**, $\text{inflow}(v)$. The sum of capacities of arcs going out is the **out-flow**, $\text{outflow}(v)$.

**Example** Below we have $\text{inflow}(v) = \text{outflow}(v) = $
Flows

Definition

Let $D$ be a network with exactly one source $A$ and one sink $Z$. Then a flow is a function that assigns a flow in $a$, $\phi(a) \geq 0$ to every $a \in A(D)$ so

1. $\phi(a) \leq c(a)$
2. inflow($v$) = outflow($v$) for all $v \in V(D)$ and $v \neq A, Z$.

Note: $\phi(a) = 0 \ \forall a$ is a zero-flow; $\phi(a) = c(a)$ is a saturated arc.

Example
**Flow value**

**Definition**

Let $D$ be a network with exactly one source $A$, one sink $Z$, and a flow $\phi$. We call

$$\text{inflow}(Z) = \text{outflow}(A)$$

the flow value.

**Goal:** We want to maximize this - find the maximum flow.

**Example** Below we have flow value = \[ \text{max. flow value} = \]

![Diagram](image-url)
**Cuts**

**Definition**

Let $D$ be a network with exactly one source $A$ and one sink $Z$. A set of arcs $S$ whose deletion disconnects $A$ and $Z$ is a cut. The sum of capacities of $s \in S$ going from component with $A$ to component with $Z$ is capacity of cut.

**Goal:** We want to minimize this - find the **minimum cut**.

**Example**
Theorem

Let $D$ be a network with exactly one source $A$ and one sink $Z$.

The maximum flow value $= \text{The capacity of a minimum cut.}$

Note: Find any flow $=$ any cut to answer. Proof is about 45 mins!

Example max. flow $=$ $= \text{min. cut.}$
Algorithm for longest path

Let $D$ be an acyclic network with source $A$ and sink $Z$ and $\ell(v)$ denote the label of $v$. Compute the longest distance from vertices $A$ to $Z$.

- Let $\ell(A) = 0$. Make it permanent. Choose a vertex $v$ all of whose arcs coming in have permanent labels $v_1, \ldots, v_k$.

Note: Permanent labels cannot be changed.

Example
Algorithm for longest path

Let $D$ be an acyclic network with source $A$ and sink $Z$ and $\ell(v)$ denote the label of $v$. Compute the longest distance from vertices $A$ to $Z$.

1. Find $\max\{\ell(v_i) + d_i\}_{i=1}^k$ where $v_i \rightarrow v$ and make this permanent.
2. Repeat until all vertices have permanent labels.

Note: Permanent labels cannot be changed.

Example
Algorithm for longest path

Let $D$ be an acyclic network with source $A$ and sink $Z$ and $\ell(v)$ denote the label of $v$. Compute the longest distance from vertices $A$ to $Z$.

- The longest distance from $A$ to $Z$ is label at $Z$. The longest path found by starting at $Z$, include arc capacity $d$ between $v, w$ if

$$\ell(w) - \ell(v) = d.$$

Example Longest distance = Longest path =
What we would normally do final class...

This can take all class and is a fun application of our theory:
In summary

- Studied networks
- Studied flows and maximum flows
- Studied cuts in networks
- Max-flow min-cut theorem
- Algorithm for longest path

Thanks for a great Math 442!