Homework 1 - due January 9th

1. Prove that $2^{2n} \geq n^4$ for all integers $n \geq 4$.

2. Prove that $x - y$ divides $x^n - y^n$ for all integers $n \geq 1$.

3. Prove that for every odd number $n \geq 1$, we have that 9 divides $4^n + 5^n$.

4. Prove that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ for all integers $n \geq 1$.

5. Prove that for every positive integer $n$, one of the numbers $n, n + 1, n + 2, \ldots, 2n$ is the square of an integer.

6. A composition of a natural number $n$ is an ordered list of positive integers whose sum is $n$. Let $c(n)$ be the number of compositions of $n$. Conjecture and then prove a formula for $c(n)$ for all $n \geq 1$.

Homework 2 - due January 16th

7. Show it is possible to traverse all the bridges in Königsburg exactly twice, returning to the starting point afterwards.

8. The local council in Königsburg decide to build two new bridges. Give one example of where they can be built so that the citizens can find a route through the town crossing each bridge only once and finishing up where they started. Show both the new bridges and the route.

9. Show that a knight can tour each square on a $3 \times 4$ chessboard – though without finishing at the starting square.

Explain why on a chessboard with an odd number of squares one can never find a knight’s tour starting and finishing at the same place.

10. How many different paths are there around an eight sided die, whose “adjacent face” graph is below, that start and end at 1 and visit each other side only once? Explain your answer.
11. Prove the game of sprouts starting with \( n \) vertices must terminate after at most \( 3n - 1 \) moves.

Write down the end-position of the game of sprouts with 3 vertices that terminated after 6 moves.

12. In a party of 6 people is it true that either there exists 4 people who all do know each other or there exists 4 people who all do not know each other? Justify your answer.

Homework 3 - due January 23rd

13. Prove that in any graph the sum of the degrees of all the vertices is even. Deduce that an even number of vertices have odd degree.

In a class with 25 students each student sends a valentine to 5 other students. Is it possible for every student to receive a valentine from exactly the 5 students they sent a valentine to? Explain your answer.

14. If a simple graph is disconnected, then is its complement connected? Explain your answer.

15. A simple graph that is isomorphic to its complement is self-complementary. Prove that if \( G \) is self-complementary then \( G \) has \( 4k \) or \( 4k + 1 \) vertices, where \( k \) is an integer.

Find all self-complementary graphs with 4 and 5 vertices.

16. Prove that every self-complementary graph with \( 4n + 1 \) vertices has at least one vertex of degree \( 2n \).

17. For each of the following graphs find the degrees of the vertices. Deduce that although all have the same number of vertices and edges, only exactly one pair of them is isomorphic. Find an isomorphism between them.
18. Let $Q_k$ be the graph whose vertices correspond to the sequences $(a_1, a_2, \ldots, a_k)$ where each $a_i = 0$ or 1, and whose edges join those sequences that differ in just one place e.g. $Q_3$ is shown below.

![Graph $Q_3$](image)

Find and prove a formula for the number of vertices of $Q_k$. Find and prove a formula for the number of edges of $Q_k$.

**Homework 4 - due January 30th**

19. Give an example of a connected graph that is
   (a) Hamiltonian but not Eulerian
   (b) Eulerian but not Hamiltonian.

   Explain your answer in each case.

20. Does there exist a simple graph that has a Hamiltonian cycle and an Eulerian trail and a vertex $v$ with $\deg(v) \geq 3$?

   If yes then give an example, and if no then prove why not.

21. Let $G$ be a connected graph and let $n$ be the length of the longest trail. Prove that if 2 trails in $G$ have length $n$, then there must be some vertex $v \in V(G)$ that is contained in both trails.

22. A mouse eats its way through a $3 \times 3 \times 3$ cube of cheese by eating all the $1 \times 1 \times 1$ subcubes. If it starts at a corner and always moves to an adjacent subcube (sharing a face of area 1) can it do this and eat the centre subcube last? If yes then give an order of subcubes eaten, if no then prove it is impossible.

23. Prove that $Q_k$ for all $k \geq 2$ is Hamiltonian.
24. Show the following four cubes problem has no solution.

Homework 5 - due February 6th

25. Prove that if a graph has no closed paths of odd length then it is bipartite.

26. If a simple connected planar graph consists of 8 vertices of degree 4 then how many faces does it have? Give an example of such a graph.

27. For exactly which values of $k$ is $Q_k$ planar? Explain your answer.

28. Find a simple graph $G$ with 8 vertices such that $G$ and its complement are both planar.

29. Prove that the average degree of vertices in a connected planar simple graph with $v > 2$ is strictly less than 6.

30. Let $G$ be a connected, planar, simple graph with $v$ vertices, and $e$ edges. Prove that if every face is isomorphic to $C_k$, $k \geq 3$ then

$$e = \frac{k(v - 2)}{k - 2}.$$

Homework 6 - due February 27th

31. Let $G_1$ and $G_2$ be two homeomorphic graphs. Let $G_1$ have $n_1$ vertices and $m_1$ edges, and let $G_2$ have $n_2$ vertices and $m_2$ edges. Show that $m_1 - n_1 = m_2 - n_2$.

32. Let $G$ be a polyhedral graph. By considering the number of edges that meet at a vertex prove that $f \geq 2 + \frac{1}{3}e$.

A polyhedron has all its faces either pentagons or hexagons. Show it must have at least 12 pentagonal faces. If exactly 3 faces meet at each vertex then prove it must have exactly 12 pentagonal faces.
33. Given a planar embedding $\tilde{G}$ of a graph $G$ the dual of $G$, denoted $G^*$, is the graph whose vertices are in one-to-one correspondence with the faces of $\tilde{G}$. Two vertices in $G^*$ are adjacent if and only if the corresponding faces in $\tilde{G}$ share an edge, and an edge is generated between 2 vertices in $G^*$ for every edge the corresponding faces share in $\tilde{G}$.

Find the dual of each of the regular polyhedra.

34. The line graph $L(G)$ of a simple graph $G$ is the graph whose vertices are in one-to-one correspondence with the edges of $G$, and two vertices in $L(G)$ are adjacent if and only if the corresponding edges in $G$ meet at a vertex.

- Show the line graph of the tetrahedron graph is the octahedron graph.
- Prove that if a simple graph $G$ is regular of degree $k > 0$, then $L(G)$ is regular of degree $2k - 2$.

35. Find two simple graphs $G$ and $H$ such that $G$ and $H$ are not isomorphic but $L(G)$ and $L(H)$ are isomorphic.

36. An induced subgraph of a graph $G$ is formed by deleting a subset of its vertices only. Prove that if $K_{1,3}$ is an induced subgraph of a graph $G$, then $G \neq L(H)$ for any graph $H$.

Homework 7 - due March 5th

37. (Without using Brooks’ Theorem!) Prove any graph without loops where all the vertices have degree $\leq 3$ is 4-colourable. Give an example of a plane graph in which no 4 vertices are all adjacent (that is, it does not contain $K_4$ as a subgraph), but which is 4-chromatic.

38. Calculate the chromatic polynomials of the two graphs below. Write your answer as a product of factors.

39. The windmill graph $Wd(n, N)$ on $N(n - 1) + 1$ vertices is the connected simple graph formed by taking $N$ copies of $K_n$ and joining them at a common vertex. Some examples are below.
Prove the chromatic polynomial of the windmill graph $Wd(n, N)$ is $k \prod_{i=1}^{n-1} (k - i)^N$.

40. Let $G$ be a simple graph with $n$ vertices. Prove that the coefficient in $P_G(k)$ of $k^n$ is 1 and of $k^{n-1}$ is $-|E(G)|$.

41. Let $G$ be a simple graph. Prove that the chromatic polynomial $P_G(k)$ is the product of the chromatic polynomials of its components.

42. For a simple connected graph $G$ with $n$ vertices, prove that $\chi(G) = n$ if and only if $G = K_n$.

**Homework 8 - due March 12th**

43. Let $G$ be a simple graph with at least one edge. Then prove that the sum of the coefficients of $P_G(k)$ is 0.

44. Try to prove the four colour theorem by adapting the proof of the five colour theorem from class. At what point does the proof fail?

45. A graph $G$ is **$k$-critical** if $\chi(G) = k$ and the deletion of any vertex yields a graph with a smaller chromatic number. Prove that if $G$ is $k$-critical then every vertex has degree at least $k - 1$.

46. Give an example of a graph that is all three of 3-chromatic, 3-chromatic(e) and 3-chromatic(f). Explain why.

47. Give and prove an explicit edge colouring of $Q_k$ with $k$ colours, and hence prove that $\chi'(Q_k) = k$.

48. Let $G$ be a simple graph with an odd number of vertices. Prove that if $G$ is regular of degree $d \geq 2$ then $\chi'(G) = d + 1$. 
Homework 9 - due March 19th

49. A lecture timetable is to be drawn up. Certain lectures must not coincide. The *s in
the following table show which lectures must not coincide. How many lecture periods are
needed to timetable all 7 lectures?

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50. Five teams play in a tournament, each playing the other four teams once. If two pitches
are available, how many periods are needed to schedule all the matches?

51. Prove a simple connected graph \( T \) is a tree if and only if \( v = e + 1 \).

52. Prove a simple connected graph \( T \) is a tree if and only if adding an edge between two
existing vertices of \( T \) creates exactly one cycle.

53. Let \( T \) be a tree with at least two vertices. Prove that \( T \) has at least two leaves.

54. Let \( T \) be a tree. Prove that \( T \) has all vertices of odd degree if and only if for every edge
\( e \) in \( T \) there is an odd number of vertices in both components of \( T - e \).

Homework 10 - due March 26th

55. Let \( T \) be a tree with average degree \( a \). Find and prove a formula for the number of
vertices in \( T \) in terms of \( a \).

56. Let \( T \) be a tree with at least two vertices and with no vertices of degree 2. Prove that
\( T \) has more leaves than non-leaf vertices.

57. Prove that all trees are bipartite. (Hint: what is the chromatic polynomial?)

58. Prove that a forest with \( n \) vertices and \( k < n \) edges has \( n - k \) connected components.

59. If \( G \) is a simple connected graph then prove that for any edge \( e \) in \( G \) the number of
spanning trees of \( G \), \( \tau(G) \), satisfies

\[
\tau(G) = \tau(G - e) + \tau(G/e).
\]
60. In how many non-isomorphic ways can the following graph be labelled? Explain your answer.

![Graph Image]

**Homework 11 - due April 2nd**

61. (a) Write down the Prüfer sequence associated with the labelled graph below.

![Prüfer Sequence Image]

(b) Draw the labelled graph associated with the Prüfer sequence (1, 2, 2, 1, 4, 5, 5).

Prove that a vertex in a labelled graph has degree $k$ if and only if its label appears $k - 1$ times in the Prüfer sequence of the graph.

62. Let integer weights be assigned to the edges of $K_n$, $n \geq 3$. Prove that the total sum of the weights on every cycle is even if and only if the total sum of the weights on every triangle (i.e. cycle of length 3) is even.

63. Find the shortest distance from vertex $A$ to each vertex in the following weighted graph. Show all your working.

![Weighted Graph Image]
64. Let $G$ be a weighted connected graph with distinct edge weights. Prove $G$ has only one minimal spanning tree.

Find a minimal spanning tree for the following.

65. For each $n \geq 1$ answer true or false: Every simple directed graph on $n$ vertices contains 2 vertices with the same indegree or 2 vertices with the same outdegree. Explain your answer in every case.

66. Prove that if a directed graph $D$ contains no directed cycles, then $D$ contains at least one source and at least one sink.

Hence or otherwise prove that if $D$ is an $n$ vertex directed graph with no directed cycles, then the vertices of $D$ can be ordered such that if an arc goes from vertex $i$ to vertex $j$ then $i < j$ for all $1 \leq i, j \leq n$. 