Homework 1 - due January 9th

1. Prove that $2^{2n} \geq n^4$ for all $n \geq 4$.

2. Prove that $x - y$ divides $x^n - y^n$ for all $n \geq 1$.

3. Prove that for every odd number $n \geq 1$, we have that 9 divides $4^n + 5^n$.

4. Prove that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ for all $n \geq 1$.

5. Prove that for every positive integer $n$, one of the numbers $n, n+1, n+2, \ldots, 2n$ is the square of an integer.

6. A composition of a natural number $n$ is an ordered list of positive integers whose sum is $n$. Let $c(n)$ be the number of compositions of $n$. Conjecture and then prove a formula for $c(n)$ for all $n \geq 1$.

Homework 2 - due January 16th

7. Show it is possible to traverse all the bridges in Kónigsburg exactly twice, returning to the starting point afterwards.

8. The local council in Kónigsburg decide to build two new bridges. Give one example of where they can be built so that the citizens can find a route through the town crossing each bridge only once and finishing up where they started. Show both the new bridges and the route.

9. Show that a knight can tour each square on a $3 \times 4$ chessboard – though without finishing at the starting square.

Explain why on a chessboard with an odd number of squares one can never find a knight’s tour starting and finishing at the same place.

10. How many different paths are there around an eight sided die, whose “adjacent face” graph is below, that start and end at 1 and visit each other side only once? Explain your answer.
11. Prove the game of sprouts starting with $n$ vertices must terminate after at most $3n - 1$ moves.

Write down the end-position of the game of sprouts with 3 vertices that terminated after 6 moves.

12. In a party of 6 people is it true that either there exists 4 people who all do know each other or there exists 4 people who all do not know each other? Justify your answer.

**Homework 3 - due January 23rd**

13. Prove that in any graph the sum of the degrees of all the vertices is even. Deduce that an even number of vertices have odd degree.

In a class with 25 students each student sends a valentine to 5 other students. Is it possible for every student to receive a valentine from exactly the 5 students they sent a valentine to? Explain your answer.

14. If a graph is disconnected, then is its complement connected? Explain your answer.

15. A simple graph that is isomorphic to its complement is *self-complementary*. Prove that if $G$ is self-complementary then $G$ has $4k$ or $4k + 1$ vertices, where $k$ is an integer.

Find all self-complementary graphs with 4 and 5 vertices.

16. Prove that every self-complementary graph with $4n + 1$ vertices has at least one vertex of degree $2n$.

17. For each of the following graphs find the degrees of the vertices. Deduce that although all have the same number of vertices and edges, only *exactly* one pair of them is isomorphic. Find an isomorphism between them.
18. Let $Q_k$ be the graph whose vertices correspond to the sequences $(a_1, a_2, \ldots, a_k)$ where each $a_i = 0$ or $1$, and whose edges join those sequences that differ in just one place e.g. $Q_3$ is shown below.

Find and prove a formula for the number of vertices of $Q_k$. Find and prove a formula for the number of edges of $Q_k$. 