1. We will do a strong induction on the number of edges $E$.

Base case: $E = 1$. Let $G$ be a simple graph with $n$ vertices where $n \geq 2$ and one edge $e$. Then by deletion-contraction and Proposition 6 we have

$$P_G(k) = P_{N_n}(k) - P_{N_{n-1}}(k) = k^n - k^{n-1}$$

and the sum of the coefficients of $P_G(k)$ is 0.

Induction step: Now assume the result is true for a graph with $E$ edges where $1 < E < m$ and consider a simple graph $G$ with $m$ edges. Then by deletion-contraction

$$P_G(k) = P_{G-e}(k) - P_{G/e}(k)$$

and by the induction assumption the sum of the coefficients of $P_{G-e}(k)$ and $P_{G/e}(k)$ each sum to 0 and hence the coefficients of $P_G(k)$ sum to 0, and the result follows by induction.

2. At the point where we delete and reinsert a vertex of degree 5 in the five colour theorem we rely on the fact that $K_5$ is not planar. However, when we adjust the proof to delete and reinsert a vertex of degree 4 in proving the four colour theorem we would need that $K_4$ is not planar, but this is not true.

3. Let $\chi(G) = k$ and the deletion of any vertex $v$ yields a graph with a smaller chromatic number, i.e. $\chi(G - v) = m \leq k - 1$. If there exists a vertex $\tilde{v} \in V(G)$ of degree less than $k - 1$ delete it. Then colour the remaining graph in $k - 1$ colours. Reinsert $\tilde{v}$. As it is adjacent to at most $k - 2$ vertices there is at least one of the $k - 1$ colours we could colour it, so $\chi(G) \leq k - 1$. Since this is not true, the degree of every vertex must be at least $k - 1$.

4. At least 3 colours are needed to colour the vertices due to the $K_3$ subgraph, at least 3 colours are needed to colour the edges due to the vertices of degree three, and at least 3 colours are needed to colour the three faces that all adjacent to each other. The diagram below shows 3 colours is also enough for each of these.
5. Recall that in $Q_k$ a vertex labelled by $a_1 \ldots a_k$ where $a_i = 0,1$ for all $1 \leq i \leq k$ is adjacent to $k$ vertices since there are $k$ choices for where a sequence can differ from $a_1 \ldots a_k$ in just one place.

We can colour the edges of $Q_k$ with $k$ colours in the following way. If two adjacent vertices have their labels differing in position $i$ where $1 \leq i \leq k$ then colour the edge between them colour $i$. Note that by construction every edge incident at a vertex is thus a different colour, and hence we have an edge colouring of $Q_k$ with $k$ colours.

Consequently, $\chi'(Q_k) \leq k$. However, since $k$ edges meet at a vertex, $\chi'(Q_k) \geq k$. Therefore $\chi'(Q_k) = k$.

6. We have no loops as the graph is simple, so it is edge colourable. Assume it is edge colourable in $d \geq 2$ colours for our $d$ regular graph.

We know that since $G$ is regular $vd = 2e$, so $v/2 = e/d$ so $e/d$ is not an integer since $v$ is odd. On the other hand since each vertex has $d$ edges incident to it, coloured $d$ colours, this means there are the same number of edges of each colour, say $p$. So $pd = e$ and $e/d$ is an integer. We have a contradiction. Thus by Vizing’s Theorem, the chromatic index is $d + 1$. 