1. We’ll prove this by weak induction.

Base case: $2^8 = 256 \geq 256 = 4^4$.

Induction step: Now assume the result is true for $n = k$, that is, $2^{2k} \geq k^4$. For $n = k + 1$
\[ (k + 1)^4 = k^4 + 4k^3 + 6k^2 + (4k + 1) \leq k^4 + k^4 + k^4 + k^4 = 4k^4 \leq 2^2(2^{2k}) = 2^{2(k+1)} \]
since $k \geq 4$, and the result follows by induction.

2. We’ll prove this by weak induction.

Base case: For $n = 1$, clearly $x - y$ divides $x - y$.

Induction step: Now assume the result is true for $n = k$, that is, $x - y$ divides $x^k - y^k$. For $n = k + 1$
\[ x^{k+1} - y^{k+1} = x^{k+1} - x y^k + x y^k - y^{k+1} = x(x^k - y^k) + (x - y)y^k. \]

By the induction assumption $x - y$ divides the RHS so also divides the LHS, and the result follows by induction.

3. We’ll prove this by weak induction.

Base case: $n = 1$ and 9 divides $4 + 5$.

Induction step: Now assume the result is true for $n = k$, that is, 9 divides $4^k + 5^k$. For $n = k + 2$ (since $n$ must stay odd)
\[ 4^{k+2} + 5^{k+2} = 16(4^k + 5^k) + 9.5^k. \]

By the induction assumption 9 divides the RHS and so also the LHS. The result now follows by induction.

4. We’ll prove this by weak induction.

Base case: $n = 1$ and $\sum_{i=1}^{1} i = 1 = \frac{1(1+1)}{2}$.

Induction step: Now assume the result is true for $n = k$, that is, $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$. For $n = k + 1$
\[ \sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k + 1) = \frac{k(k+1)}{2} + (k+1) = \frac{k^2 + 3k + 2}{2} = \frac{(k+1)(k+2)}{2} \]
and the result follows by induction.
5. We’ll prove this by weak induction.

Note for \( n = 1 \) that \( \{1, 2\} \) contains the square 1. Now we start our induction at \( n = 2 \) for convenience.

Base case: For \( n = 2 \), \( \{2, 3, 4\} \) contains the square 4.

Induction step: Now assume the result is true for \( n = k \), that is, the set \( \{k, k + 1, \ldots , 2k\} \) contains a square.
For \( n = k + 1 \) consider the set \( \{k + 1, \ldots , 2k + 2\} \). By the induction assumption if any of \( \{k + 1, \ldots , 2k\} \) is a square then we are done.
If not then the induction assumption implies that \( k \) is a square, say \( k = l^2 \). Hence

\[(l + 1)^2 = l^2 + 2l + 1 = k + 2l + 1\]

but \( 2l + 1 \leq k + 1 \) since \( k \geq 2 \) so \( (l + 1)^2 \in \{k + 1, \ldots , 2k + 2\} \). Thus a square always lies in \( \{k + 1, \ldots , 2k + 2\} \) and the result follows by induction.

6. First we’ll gather some data: \( c(1) = 1, c(2) = 2, c(3) = 4, c(4) = 8 \). From this we get the following.

**We have that** \( c(n) = 2^{n-1} \) **for all** \( n \geq 1 \).

*Proof.* We’ll prove this by weak induction.

Base case: \( n = 1 \) and \( c(1) = 1 = 2^0 \).

Induction step: Now assume the result is true for \( n = k \). For \( n = k + 1 \) note that we can create every composition of \( k + 1 \) from a composition \( a + b + \cdots + c \) of \( k \), by either \( a + b + \cdots + c + 1 \) or \( a + b + \cdots + (c + 1) \). So

\[c(k + 1) = 2c(k) = 22^{k-1} = 2^k\]

where we use the induction assumption for the second equality, and the result follows by induction. \( \Box \)