Homework 1 - due January 4th

1. Prove that $2^{2n} \geq n^4$ for all $n \geq 4$.

2. Prove that $x - y$ divides $x^n - y^n$ for all $n \geq 1$.

3. Prove that for every odd number $n \geq 1$, we have that $9$ divides $4^n + 5^n$.

4. Prove that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ for all $n \geq 1$.

5. Prove that for every positive integer $n$, one of the numbers $n, n+1, n+2, \ldots, 2n$ is the square of an integer.

6. A composition of a natural number $n$ is an ordered list of positive integers whose sum is $n$. Let $c(n)$ be the number of compositions of $n$. Conjecture and then prove a formula for $c(n)$ for all $n \geq 1$.

Homework 2 - due January 11th

7. The local council in Königsberg eventually decide to demolish one bridge. Does there exist a bridge they can demolish so the citizens can find a route through the town crossing each bridge only once and not finish up where they started? Explain your answer.

8. Show that a knight can tour each square on a $3 \times 4$ chessboard – though without finishing at the starting square.

Explain why on a chessboard with an odd number of squares $> 1$ one can never find a knight’s tour starting and finishing at the same place.

9. Write down the expression given by the following parse tree.

10. Prove the game of sprouts starting with $n$ vertices must terminate after at most $3n - 1$ moves.

Write down the end-position of the game of sprouts with 3 vertices that terminated after 6 moves.
11. In a party of 6 people is it true that either there exists 4 people who all do know each other or there exists 4 people who all do not know each other? Justify your answer.

12. Prove that every simple graph with at least two vertices contains 2 vertices with the same degree.

Homework 3 - due January 18th

13. Prove that in any graph the sum of the degrees of all the vertices is even. Deduce that an even number of vertices have odd degree.

If I am at a party and have shaken hands with 5 people prove that someone else at the party has shaken hands with an odd number of people.

14. Is it possible for a graph with an even number of vertices to have half its vertices of degree \(a\) and half its vertices of degree \(a+1\)? If it is then find and prove a necessary condition on the number of vertices and then give an example that is both simple and connected. If it is not then prove that no such graph exists.

15. A simple graph that is isomorphic to its complement is self-complementary. Prove that if \(G\) is self-complementary then \(G\) has \(4k\) or \(4k+1\) vertices, where \(k\) is an integer.

Find a connected self-complementary graph with 8 vertices.

16. Prove that no simple graph with 2 or 3 vertices is self-complementary without using Q15.

17. For each of the following graphs find the degrees of the vertices. Deduce that although all have the same number of vertices and edges, only exactly one pair of them is isomorphic. Find an isomorphism between them.
18. Let $Q_k$ be the graph whose vertices correspond to the sequences $(a_1, a_2, \ldots, a_k)$ where each $a_i = 0$ or $1$, and whose edges join those sequences that differ in just one place e.g. $Q_3$ is shown above.

Find and prove a formula for the number of vertices of $Q_k$. Find and prove a formula for the number of edges of $Q_k$.

**Homework 4 - due January 25th**

19. Show the Grötzsch graph below is Hamiltonian.

20. (a) For which values of $n$ is the complete graph $K_n$ Eulerian? Hamiltonian?

(b) For which values of $m, n$ is the complete bipartite graph $K_{m,n}$ Eulerian? Hamiltonian? Prove each of your answers.

21. (a) Does every Eulerian bipartite graph have an even number of edges?

(b) Does every Eulerian simple graph with an even number of vertices have an even number of edges?

If yes then give a proof, and if no then give a counterexample.

22. Let $G$ be a Hamiltonian graph and let $S$ be any set of $k$ vertices in $G$. Prove that the graph $G - S$ has at most $k$ components.

23. Prove that $Q_k$ for all $k \geq 2$ is Hamiltonian.
24. Show the following four cubes problem has no solution.

![Cube Diagrams](image)

**Homework 5 - due February 1st**

25. Prove that if a graph has no closed paths of odd length then it is bipartite.

26. If a simple connected planar graph consists of 5 vertices of degree 4, and 4 vertices of degree 3, then how many faces does it have? Give an example of such a graph.

27. For exactly which values of \( k \) is \( Q_k \) planar? Prove your answer.

28. Prove that every simple connected planar graph with fewer than 12 vertices contains a vertex of degree at most 4.

29. Let \( G \) be a simple graph with at least 11 vertices. Prove that \( G \) and \( \overline{G} \) are not both planar.

30. Which of the two graphs below is planar? For the one that is give a planar embedding. For the one that isn’t find a subgraph homeomorphic to \( K_5 \) or \( K_{3,3} \).

**Homework 6 - due February 15th**

31. Let \( G_1 \) and \( G_2 \) be two homeomorphic graphs. Let \( G_1 \) have \( n_1 \) vertices and \( m_1 \) edges, and let \( G_2 \) have \( n_2 \) vertices and \( m_2 \) edges. Show that \( m_1 - n_1 = m_2 - n_2 \).
32. An equivalent definition of a polyhedral graph is that it is a simple connected planar graph where every vertex has degree at least 3.

Prove that no polyhedral graph with exactly 24 edges and 8 faces can exist.

33. Given a planar embedding $\tilde{G}$ of a graph $G$ the dual of $G$, denoted $G^*$, is the graph whose vertices are in one-to-one correspondence with the faces of $\tilde{G}$. Two vertices in $G^*$ are adjacent if and only if the corresponding faces in $\tilde{G}$ share an edge, and an edge is generated between 2 vertices in $G^*$ for every edge the corresponding faces share in $\tilde{G}$.

Find the dual of each of the regular polyhedra.

34. The line graph $L(G)$ of a simple graph $G$ is the graph whose vertices are in one-to-one correspondence with the edges of $G$, and two vertices in $L(G)$ are adjacent if and only if the corresponding edges in $G$ meet at a vertex.

- Show the line graph of the tetrahedron graph is the octahedron graph.
- Prove that if a simple graph $G$ is regular of degree $k > 0$, then $L(G)$ is regular of degree $2k - 2$.

35. Given a simple graph $G$ with vertices $v_1, \ldots, v_n$, prove that the number of edges in $L(G)$ is

$$\sum_{i=1}^{n} \frac{d_i(d_i - 1)}{2}$$

where $d_i$ is the degree of vertex $v_i$ for $1 \leq i \leq n$.

36. For every $n \geq 3$ find a graph $G$ whose line graph $L(G) = K_n$. Explain your answer.

Homework 7 - due March 1st

37. (Without using Brooks’ Theorem!) Prove any graph without loops where all the vertices have degree $\leq 3$ is 4-colourable. Give an example of a plane graph in which no 4 vertices are all adjacent (that is, it does not contain $K_4$ as a subgraph), but which is 4-chromatic.

38. Calculate the chromatic polynomials of the two graphs below. Write your answer as a product of factors.
39. The windmill graph $Wd(n, N)$ on $N(n - 1) + 1$ vertices is the connected simple graph formed by taking $N$ copies of $K_n$ and joining them at a common vertex. Some examples are above.

Prove the chromatic polynomial of the windmill graph $Wd(n, N)$ is $k \prod_{i=1}^{n-1} (k - i)^N$.

40. Let $G$ be a simple graph with $n$ vertices. Prove that the coefficient in $P_G(k)$ of $k^n$ is 1 and of $k^{n-1}$ is $-|E(G)|$.

41. Let $G$ be a simple graph. Prove that the chromatic polynomial $P_G(k)$ is the product of the chromatic polynomials of its components.

42. For a simple connected graph $G$ with $n$ vertices, prove that $\chi(G) = n$ if and only if $G = K_n$. 