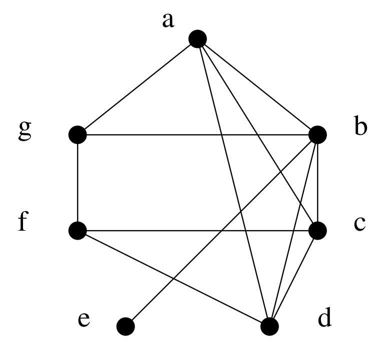
1. Drawing a graph with a vertex for each lecture, and an edge between them if they must not coincide we get the following.



Notice there is a K_4 subgraph so we need at least 4 colours. If we colour a and e colour 1, b and f colour 2, c and g colour 3 and d colour 4 we need at most 4 colours, so 4 periods are needed.

Alternatively, using deletion-contraction, we can compute the chromatic polynomial to be $k(k-1)^2(k-2)(k-3)(k^2-5k+8)$ again giving that 4 colours and hence periods are needed.

2. Draw a graph with a vertex for each team and an edge between them if they are playing a match. Now colour an edge i if the match is being played in period i. This is equivalent to finding the chromatic index of K_5 which we know is 5 by Theorem 5. Hence 5 periods are needed and since 2 edges are coloured each colour and we have 2 pitches we are done.

3. If a simple connected graph is a tree then $f = 1$ and by Euler's Theorem

$$v - e + f = 2 \Rightarrow v - e + 1 = 2 \Rightarrow v = e + 1.$$

Conversely, if a simple connected graph has $v = e + 1$ then if we assume we have a cycle containing m vertices it must contain m edges. Consequently we have $v - m$ vertices to connect to the graph with $v - 1 - m$ edges, since a vertex needs at least one edge to connect it to the graph this is impossible, so there are no cycles, so by definition we have a tree.

4. If we have a tree then since it is a simple connected graph there is exactly one path between any two vertices by Theorem 7 (so no cycles). Connecting two of them them creates exactly two paths between them, that is, one cycle, as since we had a tree there were none before.
Conversely, if adding one edge creates exactly one cycle in a simple connected graph then this means we started with a simple connected graph with no cycles, that is, a tree.

5. If v is the number of vertices in T then

$$v = \frac{2}{2-a}.$$

Proof. Let v be the number of vertices in T, and e be the number of edges. By Q51 $v = e+1$, or $e = v - 1$. We also know that $2e =$ (sum of degrees) $= av$. Hence $2(v-1) = av$ and rearranging gives $v = \frac{2}{2-a}$. □

6. We will do a proof by strong induction on the number of edges E.

Base case: If $E = 1$ then T is K_2 and T has 2 leaves and 0 non-leaf vertices.

Induction step: Assume the result is true for all trees with fewer than $m-1$ edges and consider a tree T with $m-1$ edges, L leaves and $m-L$ non-leaves (since by Q51 T has m vertices in total). Delete a leaf ℓ. If deleting ℓ results in no vertices of degree 2 then by induction

$$L - 1 > m - L \Rightarrow L > m - L.$$

If no such ℓ exists, that is, deleting every leaf results in a vertex of degree 2, then this means that in T that some vertex v is adjacent to exactly 2 leaves, or $T = K_{1,3}$. If $T = K_{1,3}$ then the result follows. If not then delete both leaves v is adjacent to, resulting in v becoming a leaf. Then by induction

$$L - 2 + 1 > m - L - 1 \Rightarrow L > m - L.$$

Hence the result follows by induction.