1. We will do a strong induction on the number of edges E.

Base case: Let $E = 0$. Then $G = N_n$ for $n \geq 1$ and we know from Proposition 6 that $P_G(k) = k^n$, whose one term alternates in sign.

Induction step: Now assume that the result is true for $0 < E < m$. Now given a simple graph with $E = m$ edges and n vertices, by deletion-contraction we get

$$P_G(k) = P_{G-e}(k) - P_{G/e}(k)$$

where by induction giving us that the coefficients alternate in sign, and knowing that the coefficient of the leading term of the chromatic polynomial is 1 by Q40 we also know that

$$P_{G-e}(k) = k^n - a_{n-1}k^{n-1} + a_{n-2}k^{n-2} - \cdots + (-1)^n a_0$$

$$P_{G/e}(k) = k^{n-1} - b_{n-2}k^{n-2} + b_{n-3}k^{n-3} - \cdots + (-1)^{n-1} b_0$$

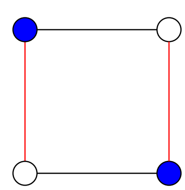
for $a_i, b_i \geq 0$ for all i. Hence

$$P_G(k) = k^n - (a_{n-1} + 1)k^{n-1} + (a_{n-2} + b_{n-2})k^{n-2} - \cdots + (-1)^n(a_0 + b_0)$$

and the result follows by induction.

2. At the point where we delete and reinsert a vertex of degree 5 in the five colour theorem we rely on the fact that K_5 is not planar. However, when we adjust the proof to delete and reinsert a vertex of degree 4 in proving the four colour theorem we would need that K_4 is not planar, but this is not true.

3. Let $\chi(G) = k$ and the deletion of any vertex v yields a graph with a smaller chromatic number, i.e. $\chi(G - v) = m \leq k - 1$. If there exists a vertex $\tilde{v} \in V(G)$ of degree less than $k - 1$ delete it. Then colour the remaining graph in $k - 1$ colours. Reinsert $\tilde{v}$. As it is adjacent to at most $k - 2$ vertices there is at least one of the $k - 1$ colours we could colour it, so $\chi(G) \leq k - 1$. Since this is not true, the degree of every vertex must be at least $k - 1$.



4.

5. Recall that in Q_k a vertex labelled by $a_1 \ldots a_k$ where $a_i = 0, 1$ for all $1 \leq i \leq k$ is adjacent to k vertices since there are k choices for where a sequence can differ from $a_1 \ldots a_k$ in just one place.

We can colour the edges of Q_k with k colours in the following way. If two adjacent vertices have their labels differing in position i where $1 \leq i \leq k$ then colour the edge between them colour i. Note that by construction every edge incident at a vertex is thus a different colour, and hence we have an edge colouring of Q_k with k colours.

Consequently, $\chi'(Q_k) \leq k$. However, since k edges meet at a vertex, $\chi'(Q_k) \geq k$. Therefore $\chi'(Q_k) = k$.

6. We have no loops as the graph is simple, so it is edge colourable. Assume it is edge colourable in $d \geq 2$ colours for our d regular graph.

We know that since G is regular $vd = 2e$, so $v/2 = e/d$ so e/d is not an integer since v is odd. On the other hand since each vertex has d edges incident to it, coloured d colours, this means there are the same number of edges of each colour, say p. So $pd = e$ and e/d is an integer. We have a contradictiction. Thus by Vizing's Theorem, the chromatic index is $d + 1$.