1. We will do a strong induction on the number of vertices \( n \).

*Base case:* \( n = 1 \). One vertex is clearly 4-colourable.

*Induction step:* Assume true for every graph with \( 1 \leq n < m \) vertices, whose vertices have degree \( \leq 3 \). Take a graph with \( m \) vertices and delete a vertex and edges joined to it. By induction colour the remaining graph with 4 colours. Insert the vertex again, and since it is adjacent to no more than 3 vertices there is a colour free to colour it with, and the graph is 4-colourable.

2. Using deletion-contraction the chromatic polynomial for the graph on the left is \( k(k-1)(k-2)(k^2-3k+3) \) and on the right is \( k(k-1)^3(k-2) \).

3. We will do a strong induction on the number of copies of \( K_n, N \).

*Base case:* \( N = 1 \). Since all the vertices in \( K_n \) are adjacent we have \( k \) choices of colour for the first vertex, \( k-1 \) for the second, \ldots, \( k-(n-1) \) for the \( n \)-th vertex. Hence

\[
P_{Wd(n,1)}(k) = P_{K_n}(k) = k(k-1) \cdots (k-(n-1)) = k \prod_{i=1}^{n-1} (k-i)^1.
\]

*Induction step:* Now assume that the result is true for \( 1 < N < m \) and consider \( Wd(n,m) \). Delete one copy of \( K_n \) from \( Wd(n,m) \) to form \( Wd(n,m-1) \). Then by induction \( P_{Wd(n,m-1)}(k) = k \prod_{i=1}^{n-1} (k-i)^{m-1} \). Inserting the one remaining copy of \( K_n \) we can colour the remaining vertices in \( (k-1)(k-2) \cdots (k-(n-1)) \) colours and hence

\[
P_{Wd(n,m)}(k) = k \prod_{i=1}^{n-1} (k-i)^{m}.
\]

4. We will first prove the coefficient of \( k^n \) is 1 AND the coefficient of \( k^{n+r} \) for \( r > 0 \) is 0. This will be a strong induction on the number of edges \( E \).

*Base case:* \( E = 0 \). \( P_G(k) = k^n \) from the notes.

*Induction step:* Assume the result is true for up to and including \( m-1 \) edges. Then for a graph \( G \) with \( m \) edges, by deletion-contraction on some \( e \in E(G) \) we have

\[
P_G(k) = P_{G-e}(k) - P_{G/e}(k).
\]
By induction the coefficient of $k^n$ is 1 in $P_{G-e}(k)$ and 0 in $P_{G/e}(k)$ and the coefficient of $k^{n+r}$ for $r > 0$ is 0, and the result follows.

We now prove the coefficient of $k^{n-1}$ is $-|E(G)|$ by strong induction on the number of edges $E$.

**Base case:** $E = 0$. $P_G(k) = k^n$ from the notes and the coefficient of $k^{n-1}$ is $0 = -|E(G)|$.

**Induction step:** Assume the result is true for up to and including $m-1$ edges. Then for a graph $G$ with $m$ edges, by deletion-contraction on some edge $e \in E(G)$ we have

$$P_G(k) = P_{G-e}(k) - P_{G/e}(k).$$

By induction the coefficient of $k^{n-1}$ in $P_{G-e}(k)$ is $-(m-1)$ and in $P_{G/e}(k)$ is 1 by the first part of this question. Hence the coefficient of $k^{n-1}$ in $P_G(k)$ is $-(m-1)-1 = -m = -|E(G)|$ as desired.

**5.** We will do a weak induction on the number of components $C$.

**Base case:** $C = 1$. $P_G(k)$ is the chromatic polynomial of the one component.

**Induction step:** Assume the result is true for a graph with $m$ components. Then for $m+1$ components, in components $C_1, \ldots, C_m$ we know that since no vertex in them is connected to any vertex in $C_{m+1}$, then $C_{m+1}$ can be coloured independently from the other components in $P_{C_{m+1}}(k)$ ways. Hence

$$P_G(k) = P_{C_1, \ldots, C_m}(k)P_{C_{m+1}}(k) = P_{C_1}(k) \cdots P_{C_m}(k)P_{C_{m+1}}(k)$$

by the induction hypothesis, and the result follows.

**6.** By Proposition 8 and the third question on this homework, we know that if $G = K_n$ then $\chi(G) = n$.

Conversely, if $G \neq K_n$ then it contains a vertex $v$ such that $\deg(v) < n-1$. Consider $G' = G - v$. Since $G'$ has $n-1$ vertices we can colour it with $n-1$ colours. Reinsert $v$ into $G'$ to recover $G$. Then since $\deg(v) < n-1$ there must be at least one of the $n-1$ colours, $c$, not used to colour a vertex adjacent to $v$. Colour $v$ with colour $c$. Hence $\chi(G) \leq n-1$, so $\chi(G) \neq n$. 