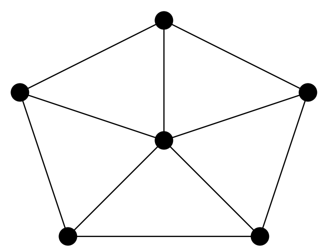
1. We will do a strong induction on the number of vertices n.

Base case: $n = 1$. One vertex is clearly 4-colourable.

Induction step: Assume true for every graph with $1 \leq n < m$ vertices, whose vertices have degree ≤ 3. Take a graph with m vertices and delete a vertex and edges joined to it. By induction colour the remaining graph with 4 colours. Insert the vertex again, and since it is adjacent to no more than 3 vertices there is a colour free to colour it with, and the graph is 4-colourable.



2. Using deletion-contraction the chromatic polynomial for the graph on the left is $k(k-1)(k-2)(k^2-3k+3)$ and on the right is $k(k-1)^3(k-2)$.

3. We will do a strong induction on the number of copies of K_n, N.

Base case: $N = 1$. Since all the vertices in K_n are adjacent we have k choices of colour for the first vertex, $k-1$ for the second, ..., $k-(n-1)$ for the n-th vertex. Hence

$$P_{Wd(n,1)}(k) = P_{K_n}(k) = k(k-1)\cdots(k-(n-1)) = k\prod_{i=1}^{n-1}(k-i)^1.$$

Induction step: Now assume that the result is true for $1 < N < m$ and consider $Wd(n,m)$. Delete one copy of K_n from $Wd(n,m)$ to form $Wd(n,m-1)$. Then by induction $P_{Wd(n,m-1)}(k) = k\prod_{i=1}^{n-1}(k-i)^{m-1}$. Inserting the one remaining copy of K_n we can colour the remaining vertices in $(k-1)(k-2)\cdots(k-(n-1))$ colours and hence

$$P_{Wd(n,m)}(k) = k\prod_{i=1}^{n-1}(k-i)^m.$$

4. We will first prove the coefficient of k^n is 1 AND the coefficient of k^{n+r} for $r > 0$ is 0. This will be a strong induction on the number of edges E.

Base case: $E = 0$. $P_G(k) = k^n$ from the notes.

Induction step: Assume the result is true for up to and including $m-1$ edges. Then for a graph G with m edges, by deletion-contraction on some $e \in E(G)$ we have

$$P_G(k) = P_{G-e}(k) - P_{G/e}(k).$$

By induction the coefficient of k^n is 1 in $P_{G-e}(k)$ and 0 in $P_{G/e}(k)$ and the coefficient of k^{n+r} for $r > 0$ is 0, and the result follows.

We now prove the coefficient of k^{n-1} is $-|E(G)|$ by strong induction on the number of edges E.

Base case: $E = 0$. $P_G(k) = k^n$ from the notes and the coefficient of k^{n-1} is $0 = -|E(G)|$.

Induction step: Assume the result is true for up to and including $m-1$ edges. Then for a graph G with m edges, by deletion-contraction on some edge $e \in E(G)$ we have

$$P_G(k) = P_{G-e}(k) - P_{G/e}(k).$$

By induction the coefficient of k^{n-1} in $P_{G-e}(k)$ is $-(m-1)$ and in $P_{G/e}(k)$ is 1 by the first part of this question. Hence the coefficient of k^{n-1} in $P_G(k)$ is $-(m-1)-1 = -m = -|E(G)|$ as desired.

5. We will do a weak induction on the number of components C.

Base case: $C = 1$. $P_G(k)$ is the chromatic polynomial of the one component.

Induction step: Assume the result is true for a graph with m components. Then for $m+1$ components, in components $\mathcal{C}_1, \ldots, \mathcal{C}_m$ we know that since no vertex in them is connected to any vertex in $\mathcal{C}_{m+1}$, then $\mathcal{C}_{m+1}$ can be coloured independently from the other components in $P_{\mathcal{C}_{m+1}}(k)$ ways. Hence

$$P_G(k) = P_{\mathcal{C}_1,\ldots,\mathcal{C}_m}(k)P_{\mathcal{C}_{m+1}}(k) = P_{\mathcal{C}_1}(k)\cdots P_{\mathcal{C}_m}(k)P_{\mathcal{C}_{m+1}}(k)$$

by the induction hypothesis, and the result follows.

6. By Proposition 8 and the third question on this homework, we know that if $G = K_n$ then $\chi(G) = n$.

Conversely, if $G \neq K_n$ then it contains a vertex v such that $deg(v) < n-1$. Consider $G' = G - v$. Since G' has $n-1$ vertices we can colour it with $n-1$ colours. Reinsert v into G' to recover G. Then since $deg(v) < n-1$ there must be at least one of the $n-1$ colours, c, not used to colour a vertex adjacent to v. Colour v with colour c. Hence $\chi(G) \leq n-1$, so $\chi(G) \neq n$.