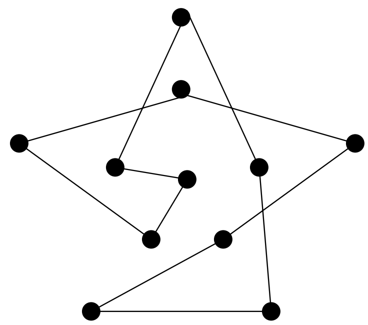


1.

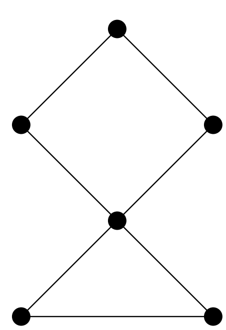
2. We know by Theorem 1 a graph is Eulerian if and only if all the vertices have even degree.

(a) K_n is Eulerian if and only if n is odd, since if n is odd then the degree of each vertex is even, whereas if n is even then the degree of each vertex is odd. It is Hamiltonian for all $n \neq 2$ as we just traverse the edges between the $n \geq 3$ vertices taken in some order as they are all connected to each other, $n = 1$ is trivially Hamiltonian from class, but $n = 2$ contains no cycle so cannot be Hamiltonian.

(b) $K_{m,n}$ is Eulerian if and only if m, n are both even, since the degree of each vertex is even, whereas if m is odd then the degree of the n vertices is odd and vice versa. It is Hamiltonian if and only if $m = n$ but $m \neq 1$ by the closed path visiting all the vertices in the order $b_1 w_1 b_2 w_2 \ldots b_m w_m b_1$. If $m \neq n$ we will have too many vertices to visit of one colour, and will not be able to visit them without visiting a vertex of the opposite colour twice.

3. (a) *Yes.* Since G is Eulerian we know by Theorem 1 that the degree of every vertex is even. Hence the degree of every white vertex is even, and so the sum of the degrees of the white vertices is even, say $2m$. Since every edge goes between a white vertex and a black vertex, $2m$ is hence the number of edges in G, which is even.

(b) *No.*



4. We first prove the result for a cycle graph C.

We'll do an induction on the number of vertices removed.

Base case $n = 1$: Consider the vertex v to be removed. Then in the natural Hamiltonian cycle in C the vertices adjacent to v are, say, v_1, v_2. When we remove v we will still have a semi-Hamitonian path on the remaining vertices starting at v_1 and ending at v_2 so the graph is still connected with 1 component.

Induction step: Now assume that the statement is true for the removal of k vertices. When we remove $k+1$ vertices, remove k of them and then note the connected component H we are removing the $k+1$-th vertex w from. Since C was originally Hamiltonian, a semi-Hamiltonian path exists in H inherited from the Hamiltonian cycle in C. Then when w is removed the semi-Hamiltonian path will become at most 2 semi-Hamiltonian paths depending on whether $deg(w) = 0, 1, 2$, and hence H will disconnect into at most 2 connected components. Thus, the number of components after the removal of $k + 1$ vertices is at most

$$\{\text{ no. of components after removal of } k \text{ vertices }\} + 1 \leq k + 1$$

and the result for the cycle graph follows by induction.

Finally, note that a Hamiltonian graph G is a cycle graph (determined by the Hamiltonian cycle) with an additional set of edges, whose existence may yet further reduce the number of connected components of G after the removal of k vertices, and the result follows.

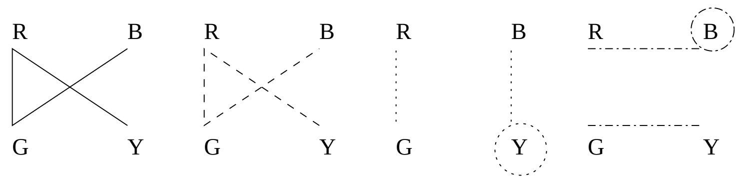
5. We'll do a strong induction on the lengths of the strings labelling the vertices.

Base case: We see $n = 2$ is Hamiltonian if we visit the vertices in the order $00, 01, 11, 10$ and then return to 00.

Induction step: Now assume that Q_k is Hamiltonian for $2 < k < m$. Then for Q_m observe that there are two distinct subgraphs of Q_m that are isomorphic to Q_{m-1}. One of these is generated by all vertices labelled by strings starting with a 1, denoted by $Q^{(1)}_{m-1}$. The other of these is generated by all vertices labelled by strings starting with a 0, denoted by $Q^{(0)}_{m-1}$. By induction each of these copies of Q_{m-1} is Hamiltonian. Hence start at the first vertex of a Hamiltonian cycle in $Q^{(1)}_{m-1}$, say $1X$ where X is a string of length $m - 1$, and go on the Hamiltonian cycle until you reach the last vertex $1Y$ where Y is a binary string of length $m - 1$. Traverse the edge from $1Y$ to $0Y$ and go on the Hamiltonian cycle backwards on $Q^{(0)}_{m-1}$ until you reach $0X$. Traverse the edge from $0X$ to $1X$ to complete the Hamiltonian cycle in Q_m.

Hence by induction Q_n for $n \geq 2$ is Hamiltonian.

6. We need 2 subgraphs such that there's one edge of each type, all vertices are degree 2, and no edge appears in both subgraphs. If we use the cube 4 loop in one of our subgraphs then we are forced to use the edge going GY in this subgraph to make a triangle between R, G and Y, but this is also on cube 4. If we use the cube 3 loop then we are forced to use the edge going RB in this subgraph. But then we cannot make a second subgraph. So this eliminates using loops, and using the other edges results in vertices of degree 3, so there is no solution.



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