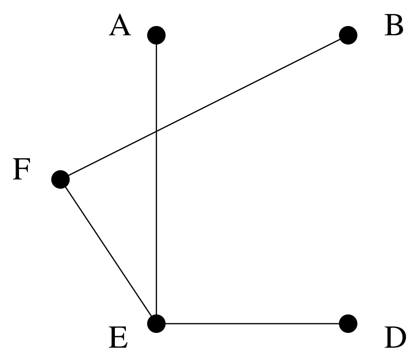
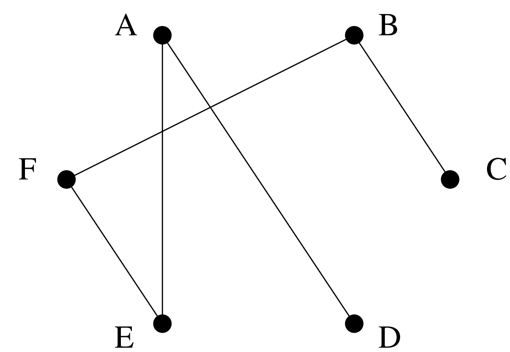
1. For the lower bound remove C and get a minimal spanning tree below of weight 6. Hence a lower bound is $6 + 4 + 4 = 14$.
For the upper bound a minimal spanning tree is as below. Starting at D we get DAEFBCD and an upper bound is 16.



lower bound



upper bound

2. If $indeg(v) = outdeg(v)$ for all $v \in V(D)$, then choose a vertex A to start at. On a directed path from A, every time we go into a vertex we can leave and eventually we arrive back at a vertex we have visited before. This forms a cycle C_1. Remove the edges of C_1 to form D_1, and observe that for each vertex in the cycle its indegree and outdegree have each reduced by one. Thus we still have $indeg(v) = outdeg(v)$ for all vertices $v \in V(D_1)$. Repeat until no arcs remain, to obtain cycles $C_1, C_2, \ldots, C_k$ which by construction have decomposed D into directed cycles.

3. Take a directed path of maximum length $v_1 \cdots v_k$, which we know exists since there are finitely many vertices. Then v_1 must be a source otherwise we could add an incoming arc to v_1 to make a longer directed path. Similarly, v_k must be a sink otherwise we could add an outgoing arc from v_k to make a longer directed path.

For the second part we will do a strong induction on the number of vertices.

Base case: The result holds with a digraph consisting of one vertex.

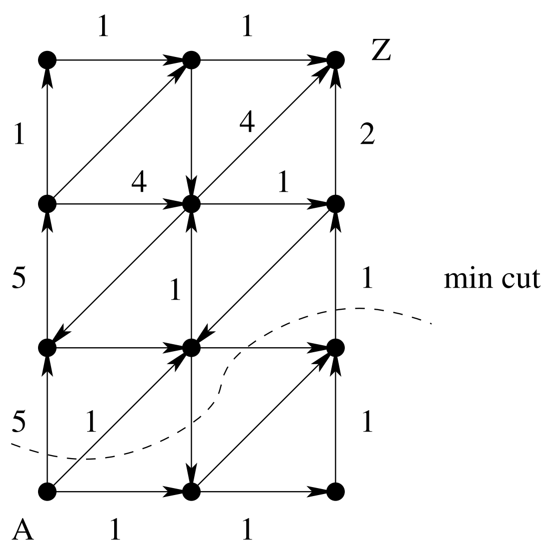
Induction step: Assume the result is true for all digraphs without directed cycles and up to $n-1$ vertices. Let D be a digraph without directed cycles with n vertices. By the first part we know D contains a sink v. Delete v to form D' (that has no directed cycles as there were none in D) and by induction label the $n-1$ vertices $1, \ldots, n-1$ in D'. Reinsert v and label it n. By construction the vertices of D are ordered such that if an arc goes from vertex i to vertex j then $i < j$ for all $1 \leq i, j \leq n$, and the result follows by induction.

4. To get from $(0, 0)$ to (m, n) we need to take m arcs parallel to the x-axis, and n parallel to the y-axis. So $m+n$ arcs in total. Note that our directed path will be completely determined by exactly which of the arcs in the directed path $a_1 a_2 \cdots a_{m+n}$ are chosen to be parallel to the x-axis, of which we have $m + n$ choices for the first arc, $m + n - 1$ choices for the second arc, ..., $m + n - m + 1$ choices for the m-th arc. So in total

$$(m + n)(m + n - 1) \cdots (n + 1) = \frac{(m + n)!}{n!}.$$

However the $m!$ different ways we could choose these m arcs does not matter, just the set of arcs themselves, so we must divide this total through by $m!$ and the result follows.

5. Checking all cuts we find the minimum cut is 7, so a max flow is



6. We will prove this is impossible by a proof by contradiction. First denote each $1 \times 1 \times 1$ subcube by its (x, y, z) coordinate, and without loss of generality assume the mouse starts at $(1, 1, 1)$ and eats its way from $(1, 1, 1)$ to $(2, 2, 2)$. Note that as it eats its way from one subcube to the next, the sum of its coordinates goes from odd to even, or even to odd. Thus as it eats its way through the 27 subcubes the sum of coordinates is odd, even, odd, even, ..., odd.
However, the sum of the coordinates of $(2, 2, 2)$ is even and we have a contradiction. So our assumption is wrong, that is, the mouse cannot eat the centre subcube last.