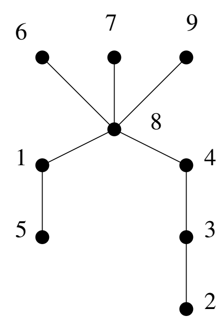
1. The total number of spanning trees $\tau(G)$ = number of spanning trees that don't include e + number of spanning trees that include e.
The former is equal to the number of spanning trees that would exist if e wasn't there, i.e $\tau(G-e)$. The latter is the number of spanning trees that would exist if the two vertices at either end of e were the same vertex (since they'll always be connected by e), i.e. $\tau(G/e)$. Hence

$$\tau(G) = \tau(G-e) + \tau(G/e).$$

(Be careful! If you use this algorithm multiple edges ARE important when you contract.)

2. Each graph has 5 vertices and hence there are 5.4.3.2.1 = 120 ways to label these vertices. However each labelled graph G is isomorphic to the labelled graph that is G reflected along the vertical axis. Thus there are $120/2 = 60$ different non-isomorphic ways to label each graph.

3. (a) $(5, 2, 2, 6, 2, 2)$.



(b)

4. If a vertex v has degree k then each time a vertex adjacent to it is removed the label of v appears in the sequence until v itself is a leaf, so its label has appeared in the list $k-1$ times. The next time an edge is removed at v is when v itself is removed, and so its label does not appear in the sequence again.
Conversely, if a label A appears $k-1$ times in the Prüfer sequence then each time the label A appears an edge is being removed from the corresponding vertex until the vertex with label A is a leaf. Then the label A does not appear again, hence the degree of the vertex is $k-1+1=k$.

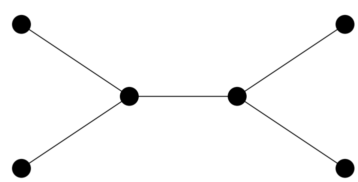
5. Assume that such a tree T exists. Then if we label T and generate its Prüfer sequence, then by Proposition 10 the Prüfer sequence of T will have length $12-2=10$. However, by Q58 we know that if a vertex is of degree k then its label appears $k-1$ times in the Prüfer sequence. Hence since T has 1 vertex of degree 5, 2 of degree 4, and 1 of degree 2, the Prüfer sequence of T has length at least

$$(5-1) + 2(4-1) + (2-1) = 11.$$

We have a contradiction, and hence no such T exists.

6. Before we begin to label the trees, we need to work out how many unlabelled trees satisfy the criteria.

Consider a tree T with 6 vertices all of which are degree 1 or 3, say m_1 of degree 1 and m_2 of degree 3. Then by Q51 we know that T has 5 edges, and summing the degrees we get $m_1 + 3m_2 = 10$. By Q54 we have $m_1 > m_2$ and since T has 5 edges $m_1 \leq 5$. Hence the only solution is $m_1 = 4, m_2 = 2$ and hence T is the tree below.



Now if we label the vertices of T and generate its Prüfer sequence, then by Proposition 10 and Q58 we know the Prüfer sequence will have length 4 with 2 numbers appearing twice in once of the following 3 ways

$$(a, a, b, b) \quad (a, b, a, b) \quad (a, b, b, a)$$

where we have 6 choices for a and 5 for b. Hence the labelling of T is one of $3.6.5 = 90$ possible labellings.

Hence 90 different labelled trees exist.