1. We'll prove this by weak induction.

Base case: $2^8 = 256 \geq 256 = 4^4$.

Induction step: Now assume the result is true for $n = k$, that is, $2^{2k} \geq k^4$. For $n = k + 1$

$$(k+1)^4 = k^4 + 4k^3 + 6k^2 + (4k+1) \leq k^4 + k^4 + k^4 + k^4 = 4k^4 \leq 2^2(2^{2k}) = 2^{2(k+1)}$$

since $k \geq 4$, and the result follows by induction.

2. We'll prove this by weak induction.

Base case: For $n = 1$, clearly $x - y$ divides $x - y$.

Induction step: Now assume the result is true for $n = k$, that is, $x - y$ divides $x^k - y^k$. For $n = k + 1$

$$\begin{aligned} x^{k+1} - y^{k+1} &= x^{k+1} - xy^k + xy^k - y^{k+1} \\ &= x(x^k - y^k) + (x-y)y^k. \end{aligned}$$

By the induction assumption $x - y$ divides the RHS so also divides the LHS, and the result follows by induction.

3. We'll prove this by weak induction.

Base case: $n = 1$ and 9 divides $4 + 5$.

Induction step: Now assume the result is true for $n = k$, that is, 9 divides $4^k + 5^k$. For $n = k + 2$ (since n must stay odd)

$$4^{k+2} + 5^{k+2} = 16(4^k + 5^k) + 9.5^k.$$

By the induction assumption 9 divides the RHS and so also the LHS. The result now follows by induction.

4. We'll prove this by weak induction.

Base case: $n = 1$ and $\sum_{i=1}^{1} i = 1 = \frac{1(1+1)}{2}$.

Induction step: Now assume the result is true for $n = k$, that is, $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$. For $n = k + 1$

$$\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{k^2+3k+2}{2} = \frac{(k+1)(k+2)}{2}$$

and the result follows by induction.

5. We'll prove this by weak induction.

Note for $n = 1$ that $\{1, 2\}$ contains the square 1. Now we start our induction at $n = 2$ for convenience.

Base case: For $n = 2$, $\{2, 3, 4\}$ contains the square 4.

Induction step: Now assume the result is true for $n = k$, that is, the set $\{k, k+1, \ldots, 2k\}$ contains a square.
For $n = k+1$ consider the set $\{k+1, \ldots, 2k+2\}$. By the induction assumption if any of $\{k+1, \ldots, 2k\}$ is a square then we are done.
If not then the induction assumption implies that k is a square, say $k = l^2$. Hence

$$(l+1)^2 = l^2 + 2l + 1 = k + 2l + 1$$

but $2l + 1 \leq k + 1$ since $k \geq 2$ so $(l+1)^2 \in \{k+1, \ldots, 2k+2\}$. Thus a square always lies in $\{k+1, \ldots, 2k+2\}$ and the result follows by induction.

6. First we'll gather some data: $c(1) = 1, c(2) = 2, c(3) = 4, c(4) = 8$. From this we get the following.

We have that $c(n) = 2^{n-1}$ for all $n \geq 1$.

Proof. We'll prove this by weak induction.

Base case: $n = 1$ and $c(1) = 1 = 2^0$.

Induction step: Now assume the result is true for $n = k$. For $n = k+1$ note that we can create every composition of $k+1$ from a composition $a + b + \cdots + c$ of k, by either $a + b + \cdots + c + 1$ or $a + b + \cdots + (c+1)$. So

$$c(k+1) = 2.c(k) = 2.2^{k-1} = 2^k$$

where we use the induction assumption for the second equality, and the result follows by induction. □