1. Prove that \( n! \geq 2^n \) for all \( n \geq 4 \).

2. Prove that \( 1 + x + \cdots + x^{n-1} + \frac{x^n}{1-x} = \frac{1}{1-x} \) for all \( n \geq 1 \).

3. Prove that \( x^n - 1 = (x-1)(x^{n-1} + \cdots + x + 1) \) for all \( n \geq 1 \).

4. Prove that 5 divides \( 3^{4n} - 1 \) for all \( n \geq 1 \).

5. Show that for any integer \( n \), the only common divisor of \( n \) and \( n + 1 \) is 1.

6. A composition of a natural number \( n \) is an ordered list of positive integers whose sum is \( n \). Let \( c(n) \) be the number of compositions of \( n \). Conjecture and then prove a formula for \( c(n) \) for all \( n \geq 1 \).

7. Show it is possible to traverse all the bridges in Königsgburg exactly twice, returning to the starting point afterwards.

8. The local council in Königsgburg decide to build two new bridges. Give one example of where they can be built so that the citizens can find a route through the town crossing each bridge only once and finishing up where they started. Show both the new bridges and the route.

9. Show that a knight can tour each square on a \( 3 \times 4 \) chessboard – though without finishing at the starting square.

Explain why on a chessboard with an odd number of squares one can never find a knight’s tour starting and finishing at the same place.

10. How many different paths are there around an eight sided die, whose “adjacent face” graph is below, that start and end at 1 and visit each other side only once? Explain your answer.

11. Prove the game of sprouts starting with \( n \) vertices must terminate after at most \( 3n - 1 \) moves.

Write down the end-position of the game of sprouts with 3 vertices that terminated after 6 moves.
12. In a party of 6 people is it true that either there exists 4 people who all do know each other or there exists 4 people who all do not know each other? Justify your answer.

**Homework 3 - due January 19th**

13. Prove that in any graph the sum of the degrees of all the vertices is even. Deduce that an even number of vertices have odd degree.

In a class with 25 students each student sends a valentine to 5 other students. Is it possible for every student to receive a valentine from exactly the 5 students they sent a valentine to? Explain your answer.

14. If a graph is disconnected, then is its complement connected? Explain your answer.

15. A simple graph that is isomorphic to its complement is *self-complementary*. Prove that if $G$ is self-complementary then $G$ has $4k$ or $4k + 1$ vertices, where $k$ is an integer.

Find all self-complementary graphs with 4 and 5 vertices.

16. Prove that every self-complementary graph with $4n + 1$ vertices has at least one vertex of degree $2n$.

17. For each of the following graphs find the degrees of the vertices. Deduce that although all have the same number of vertices and edges, only exactly one pair of them is isomorphic. Find an isomorphism between them.

18. Let $Q_k$ be the graph whose vertices correspond to the sequences $(a_1, a_2, \ldots, a_k)$ where each $a_i = 0$ or 1, and whose edges join those sequences that differ in just one place e.g. $Q_3$ is shown below.

Find and prove a formula for the number of vertices of $Q_k$. Find and prove a formula for the number of edges of $Q_k$. 
Homework 4 - due January 26th

19. If a connected graph has $k > 0$ vertices of odd degree, prove that one can find $\frac{k}{2}$ non-closed trails on the graph that together cover all the edges of the graph exactly once.

Find four non-closed trails that together cover all the edges of the following graph.

![Graph Image]

20. Can an Eulerian simple graph with an even number of vertices and odd number of edges exist? If it can then give an example, and if it cannot then explain why not.

21. Is it true that if $G$ is Eulerian with edges $e, f$ sharing a vertex $v$, whose removals leave $G$ connected, then $G$ has an Eulerian trail in which $e, f$ are consecutive? If true then give a proof, and if false then give a counter example.

22. A mouse eats its way through a $3 \times 3 \times 3$ cube of cheese by eating all the $1 \times 1 \times 1$ subcubes. If it starts at a corner and always moves to an adjacent subcube (sharing a face of area 1) can it do this and eat the centre subcube last? If yes then give an order of subcubes eaten, if no then prove it is impossible.

23. Prove that $Q_k$ for all $k \geq 2$ is Hamiltonian.

24. Show the following four cubes problem has no solution.

![Cubes Image]
Homework 5 - due February 2nd

25. Prove that if a graph has no closed paths of odd length then it is bipartite.

26. If a simple connected planar graph consists of 8 vertices of degree 4 then how many faces does it have? Give an example of such a graph.

27. For which values of $k$ is $Q_k$ planar? Explain your answer.

28. Find a graph $G$ with 8 vertices such that $G$ and its complement are both planar.

29. Prove that the average degree of vertices in a connected planar simple graph with $v > 2$ is strictly less that 6.

30. Let $G$ be a connected, planar, simple graph with $v$ vertices, and $e$ edges. Prove that if every face is isomorphic to $C_k$ then

$$e = \frac{k(v - 2)}{k - 2}.$$ 

Homework 6 - due February 16th

31. Let $G_1$ and $G_2$ be two homeomorphic graphs. Let $G_1$ have $n_1$ vertices and $m_1$ edges, and let $G_2$ have $n_2$ vertices and $m_2$ edges. Show that $m_1 - n_1 = m_2 - n_2$.

32. An equivalent definition of a polyhedral graph is that it is a simple connected planar graph where every vertex has degree at least 3.

Prove that no polyhedral graph with exactly 16 vertices and 9 faces can exist.

33. Given a planar embedding $\tilde{G}$ of a graph $G$ the dual of $G$, denoted $G^*$, is the graph whose vertices are in one-to-one correspondence with the faces of $\tilde{G}$. Two vertices in $G^*$ are adjacent if and only if the corresponding faces in $\tilde{G}$ share an edge, and an edge is generated between 2 vertices in $G^*$ for every edge the corresponding faces share in $\tilde{G}$.

Find the dual of each of the regular polyhedra.

34. The line graph $L(G)$ of a simple graph $G$ is the graph whose vertices are in one-to-one correspondence with the edges of $G$, and two vertices in $L(G)$ are adjacent if and only if the corresponding edges in $G$ meet at a vertex.

- Show the line graph of the tetrahedron graph is the octahedron graph.
- Prove that if a simple graph $G$ is regular of degree $k$, then $L(G)$ is regular of degree $2k - 2$.

35. Find two simple graphs $G$ and $H$ such that $G$ and $H$ are not isomorphic but $L(G)$ and $L(H)$ are isomorphic.

36. Prove that the line graph of a Hamiltonian simple graph with $\geq 3$ edges is Hamiltonian.
Homework 7 - due March 2nd

37. Calculate the chromatic polynomials of the two graphs below. Write your answer as a product of factors.

![Graphs](image)

38. Find the chromatic number of the Petersen graph. Justify your answer.

39. Prove that the chromatic polynomial of any tree (a connected graph that contains no cycles) with \( n \) vertices is \( k(k-1)^{n-1} \).

40. Prove the chromatic polynomial of the cycle graph \( C_n \) satisfies

\[
P_{C_n}(k) = k(k-1)^{n-1} - P_{C_{n-1}}(k).
\]

Prove the chromatic polynomial of the cycle graph \( C_n \) is \( (k-1)^n + (-1)^n(k-1) \).

41. Let \( G \) be a simple graph with \( n \) vertices. Prove that the coefficient in \( P_G(k) \) of \( k^n \) is 1 and of \( k^{n-1} \) is \(-|E(G)|\).

42. Let \( G \) be a simple graph. Prove that the chromatic polynomial \( P_G(k) \) is the product of the chromatic polynomials of its components.

Homework 8 - due March 9th

43. If \( G \) is a connected graph then is \( \chi(G) \leq 1 + av(G) \), where \( av(G) \) is the average of the vertex degrees in \( G \)? If yes then give a proof, and if no then give a counter example.

44. Try to prove the four colour theorem by adapting the proof of the five colour theorem from class. At what point does the proof fail?

45. A graph \( G \) is \( k \)-critical if \( \chi(G) = k \) and the deletion of any vertex yields a graph with a smaller chromatic number. Prove that if \( G \) is \( k \)-critical then every vertex has degree at least \( k - 1 \).

46. Give an example of a graph that is both 4-chromatic(f) and 4-chromatic.

47. Give an explicit edge colouring of \( Q_k \) with \( k \) colours, and hence prove that \( \chi'(Q_k) = k \).

48. Let \( G \) be a simple graph with an odd number of vertices. Prove that if \( G \) is regular of degree \( d \geq 2 \) then \( \chi'(G) = d + 1 \).
Homework 9 - due March 16th

49. A lecture timetable is to be drawn up. Certain lectures must not coincide. The *'s in the following table show which lectures must not coincide. How many lecture periods are needed to timetable all 7 lectures?

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
<th>g</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td></td>
<td>*</td>
<td></td>
<td></td>
</tr>
<tr>
<td>b</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td></td>
<td></td>
<td>*</td>
</tr>
<tr>
<td>c</td>
<td>*</td>
<td>*</td>
<td></td>
<td>*</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>d</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>e</td>
<td></td>
<td>*</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>f</td>
<td></td>
<td>*</td>
<td>*</td>
<td></td>
<td></td>
<td>*</td>
<td></td>
</tr>
<tr>
<td>g</td>
<td>*</td>
<td>*</td>
<td></td>
<td></td>
<td></td>
<td>*</td>
<td></td>
</tr>
</tbody>
</table>

50. Five teams play in a tournament, each playing the other four teams once. If two pitches are available, how many periods are needed to schedule all the matches?

51. Prove a simple connected graph $T$ is a tree if and only if $v = e + 1$.

52. Let $G$ be a connected graph with $n$ vertices. Prove that $G$ has exactly one cycle if and only if $G$ has exactly $n$ edges.

53. Prove that all trees are bipartite. Hence prove that every tree has a leaf in the set of vertices coloured black, or the set of vertices coloured white, whichever has the larger cardinality (or both if they have equal size).

54. Let $r_1, r_2, \ldots, r_n$ be any positive integers where $n \geq 2$. Prove that there exists a tree with $n$ vertices where vertex $v_i$ has degree $r_i$ if and only if

$$r_1 + r_2 + \cdots + r_n = 2(n - 1).$$

Draw all the nonisomorphic trees on 7 vertices that have $r_1 = 1, r_2 = 1, r_3 = 1, r_4 = 1, r_5 = 2, r_6 = 3, r_7 = 3$.

Homework 10 - due March 23rd

55. If $G$ is a simple connected graph then prove that for any edge $e$ in $G$

$$\tau(G) = \tau(G - e) + \tau(G/e).$$
56. In how many non-isomorphic ways can the following graph be labelled? Explain your answer.

![Graph Image]

57. (a) Write down the Prüfer sequence associated with the labelled graph below.

![Graph Image]

(b) Draw the labelled graph associated with the Prüfer sequence (1, 2, 2, 1, 4, 5, 5).

58. Prove that a vertex in a labelled graph has degree $k$ if and only if its label appears $k - 1$ times in the Prüfer sequence of the graph.

59. Prove that there does not exist a tree consisting of 10 vertices where 3 of the vertices have degree 4.

60. Calculate the number of vertices in a tree with 2 vertices of degree 4, 2 vertices of degree 2 and the rest leaves. Explain your answer.

Homework 11 - due March 30th

61. Prove the number of spanning trees of $K_{3,m}$ is $3^{m-1}m^2$.

62. Assign integer weights to the edges of $K_n$, $n \geq 3$. Prove that the total weight on every cycle is even if and only if the total weight on every triangle (i.e. cycle of length 3) is even.

63. Find a simple connected graph and assign the edge weights (1, 1, 2, 2, 3, 3, 4, 4) in two ways: one way so the minimal spanning tree is unique, and another way so it is not unique.

64. Starting with the vertex labelled 1, label the vertices of the following graphs using DFS then repeat using BFS given in class.
65. Find the shortest distance from vertex A to each vertex in the following weighted graph.

66. Let $G$ be a weighted connected graph with distinct edge weights. Prove $G$ has only one minimal spanning tree.

Find a minimal spanning tree for the following.
67. Find a lower bound $> 13$ and an upper bound $< 18$ for the Travelling Salesman Problem given by the following graph.

68. For each $n \geq 1$ answer true or false: Every simple digraph on $n$ vertices contains 2 vertices with the same indegree or 2 vertices with the same outdegree.

69. Prove that if a digraph $D$ contains no directed cycles, then $D$ contains at least one source and at least one sink.

Hence or otherwise prove that if $D$ is an $n$ vertex digraph with no directed cycles, then the vertices of $D$ can be ordered such that if an arc goes from vertex $i$ to vertex $j$ then $i < j$ for all $1 \leq i, j \leq n$.

70. Prove that there exists an $n$ vertex digraph whose underlying graph is $K_n$ with $\text{indeg}(v) = \text{outdeg}(v)$ at every vertex if and only if $n$ is odd.

71. Prove that a digraph is strongly connected if and only if for each partition of the set of vertices $V$ into nonempty sets $S$ and $T$ (i.e. $S \cup T = V$ and $S \cap T = \emptyset$), there is an arc from $S$ to $T$. 
72. Find a maximal flow and a minimal cut in the following network.