1. Drawing a graph with a vertex for each lecture, and an edge between them if they must not coincide we get the following.

Notice there is a $K_4$ subgraph so we need at least 4 colours. If we colour $a$ and $e$ colour 1, $b$ and $f$ colour 2, $c$ and $g$ colour 3 and $d$ colour 4 we need at most 4 colours, so 4 periods are needed.

Alternatively, using deletion-contraction, we can compute the chromatic polynomial to be $k(k-1)^2(k-2)(k-3)(k^2-5k+8)$ again giving that 4 colours and hence periods are needed.

2. Draw a graph with a vertex for each team and an edge between them if they are playing a match. Now colour an edge $i$ if the match is being played in period $i$. This is equivalent to finding the chromatic index of $K_5$ which we know is 5 by Theorem 5. Hence 5 periods are needed and since 2 edges are coloured each colour and we have 2 pitches we are done.

3. If a simple connected graph is a tree then $f = 1$ and by Euler’s Theorem

$$v - e + f = 2 \Rightarrow v - e + 1 = 2 \Rightarrow v = e + 1.$$  

Conversely, if a simple connected graph has $v = e + 1$ then if we assume we have a cycle containing $m$ vertices it must contain $m$ edges. Consequently we have $v - m$ vertices to connect to the graph with $v - 1 - m$ edges, since a vertex needs at least one edge to connect it to the graph this is impossible, so there are no cycles, so by definition we have a tree.

4. Let $G$ be a connected graph with $n$ vertices, $k$ edges and contains exactly one cycle. Let $e$ be an edge in the cycle. Then $G - e$ with $k - 1$ edges is connected (since no edge in a cycle is a bridge) and contains no cycles. Hence $G - e$ is a tree by definition. Hence by the third question this is if and only if $n = k - 1 + 1 = k$, that is $G$ has exactly $n$ edges.

5. If we have a tree $T$ with $n$ vertices, then by Q 39 its chromatic polynomial is $k(k - 1)^{n-1}$. Hence $\chi(T) = 2$ by Proposition 8 (or $\chi(N_1) = 1$), and $T$ is bipartite.

Since every tree $T$ is bipartite the vertices of $T$ can be coloured back or white such that every edge goes between a black and white vertex. We denote by $W_T$ the set of all vertices of $T$ coloured white, and denote by $B_T$ the set of all vertices coloured black.
Now we will do a proof by contradiction. Assume that there exists a tree $T$ that has been coloured such that all its leaves are in the set $W_T$ or $B_T$ with smaller cardinality. Without loss of generality, say $B_T$. Thus, $|W_T| \geq |B_T|$. Then all vertices in $W_T$ have degree at least 2, so since $T$ is bipartite and thus every edge goes between a white and black vertex

$$e \geq 2|W_T| \geq |W_T| + |B_T| = v = e + 1$$

where $e$ is the number of edges in $T$, $v$ is the number of vertices, and the last equality is by the third question. Hence our assumption is false and the result follows.

6. If we have a tree then

$$r_1 + \cdots + r_n = (\text{sum of the degrees}) = 2e = 2(n - 1)$$

by the third question.

Conversely, to have $n$ positive integers summing to $2(n - 1)$ at least 2 of them must be 1, and if $n > 2$ at least one must be $> 1$. We will now do an induction on $n \geq 2$.

Base case: For $n = 2$ we have the tree $K_2$.

Induction step: Assume that for $n = k$ the result is true. Then for $n = k + 1$, let $r_1, \ldots, r_{k+1}$ be positive integers such that

$$r_1 + \cdots + r_k + r_{k+1} = 2k.$$  

We know that one number of $r_1, \ldots, r_{k+1}$ is equal to 1, say $r_{k+1}$ and one is greater than 1, say $r_k$. Then

$$r_1 + \cdots + (r_k - 1) = 2(k - 1)$$

where $r_1, \ldots, r_k - 1$ are positive integers. So by induction there is a tree with $k$ vertices where the vertex $v_i$ has degree $r_i$ for $1 \leq i \leq k - 1$ and vertex $v_k$ has degree $r_k - 1$. Now attach a leaf $v_{k+1}$ to vertex $v_k$. Then we have created a tree with $k+1$ vertices where vertex $v_i$ has degree $r_i$ for $1 \leq i \leq k + 1$. 

![Diagram](image-url)