1. Using deletion-contraction the chromatic polynomial for the graph on the left is \( k(k - 1)(k - 2)(k^2 - 3k + 3) \) and on the right is \( k(k - 1)^3(k - 2) \).

2. Let \( P \) be the Petersen graph. The minimum number of colours required to colour any subgraph of a graph will be a lower bound for the chromatic number of the graph. Therefore any of the cycles of length 5 in \( P \), which requires at least 3 colours to colour it, gives that 

\[ 3 \leq \chi(P). \]

On the other hand, since the degree of every vertex in \( P \) is 3 we have by Brook’s Theorem 

\[ \chi(P) \leq 3. \]

Hence \( \chi(P) = 3 \).

3. Since we have a tree, start at a degree 1 vertex \( v \) and colour it one of \( k \) colours. We can colour the one vertex adjacent to it one of \( k - 1 \) colours. Each uncoloured vertex adjacent to it can be coloured one of \( k - 1 \) colours and this can be repeated until we end at another vertex of degree 1, since there are no cycles and hence a unique path between \( v \) and the vertex presently being coloured. Since we have \( n \) vertices it follows the chromatic polynomial is \( k(k - 1)^{n-1} \).

4. Using deletion-contraction and the previous question on this homework, we get 

\[ P_{C_n}(k) = k(k - 1)^{n-1} - P_{C_{n-1}}(k). \]

We will do strong induction on the number of vertices \( n \).

**Base case:** \( n = 3 \). Since \( C_3 = K_3 \), we have \( k \) choices to colour the first vertex, then \( k - 1 \) for the second, then \( k - 2 \) for the third and 

\[ P_{C_3} = k(k - 1)(k - 2) = k^3 - 3k^2 + 2k = (k - 1)^3 + (-1)^3(k - 1). \]

**Induction step:** Assume the result is true for up to and including \( m - 1 \) vertices. Then by the first part of the question

\[ P_{C_m}(k) = k(k - 1)^{m-1} - P_{C_{m-1}}(k) \]

\[ = k(k - 1)^{m-1} - (k - 1)^{m-1} - (-1)^{m-1}(k - 1) \]

\[ = (k - 1)^m + (-1)^m(k - 1). \]

5. We will first prove the coefficient of \( k^n \) is 1 AND the coefficient of \( k^{n+r} \) for \( r > 0 \) is 0. This will be a strong induction on the number of edges \( E \).

**Base case:** \( E = 0 \). \( P_G(k) = k^n \) from the notes.
**Induction step:** Assume the result is true for up to and including \( m - 1 \) edges. Then for a graph \( G \) with \( m \) edges, by deletion-contraction on some \( e \in E(G) \) we have

\[
P_G(k) = P_{G-e}(k) - P_{G/e}(k).
\]

By induction the coefficient of \( k^n \) is 1 in \( P_{G-e}(k) \) and 0 in \( P_{G/e}(k) \) and the coefficient of \( k^{n+r} \) for \( r > 0 \) is 0, and the result follows.

We now prove the coefficient of \( k^{n-1} \) is \(-|E(G)|\) by strong induction on the number of edges \( E \).

**Base case:** \( E = 0 \). \( P_G(k) = k^n \) from the notes and the coefficient of \( k^{n-1} \) is 0 = \(-|E(G)|\).

**Induction step:** Assume the result is true for up to and including \( m - 1 \) edges. Then for a graph \( G \) with \( m \) edges, by deletion-contraction on some edge \( e \in E(G) \) we have

\[
P_G(k) = P_{G-e}(k) - P_{G/e}(k).
\]

By induction the coefficient of \( k^{n-1} \) in \( P_{G-e}(k) \) is \(-(m-1)\) and in \( P_{G/e}(k) \) is 1 by the first part of this question. Hence the coefficient of \( k^{n-1} \) in \( P_G(k) \) is \(-(m-1)-1 = -m = -|E(G)|\) as desired.

6. This time we will do weak induction on the number of components \( C \).

**Base case:** \( C = 1 \). \( P_G(k) \) is the chromatic polynomial of the one component.

**Induction step:** Assume the result is true for a graph with \( m \) components. Then for \( m + 1 \) components, in components \( C_1, \ldots, C_m \) we know that since no vertex in them is connected to any vertex in \( C_{m+1} \), then \( C_{m+1} \) can be coloured independently from the other components in \( P_{C_{m+1}}(k) \) ways. Hence

\[
P_G(k) = P_{C_1,\ldots,C_m}(k)P_{C_{m+1}}(k) = P_{C_1}(k) \cdots P_{C_m}(k)P_{C_{m+1}}(k)
\]

by the induction hypothesis, and the result follows.