1. We will do an induction on the number of edges $E$. 

Base case: $E = 0$. If there are no edges then the graph is bipartite as we can colour every vertex either black or white.

Induction step: Assume every graph with $0 < k < m$ edges with no closed paths of odd length is bipartite. Now consider a graph $G$ with no closed paths of odd length and $m$ edges, and delete one of its edges $e$ whose end points are $u$ and $v$. Note that no closed paths of odd length are created so by induction $G - e$ is bipartite.

If $e$ disconnects a component of the graph into two, then by induction each component is bipartite, and we can colour the vertices such that $u$ and $v$ are different colours. If $e$ does not disconnect a component, then since $e$ completes a cycle of even length in $G$ (by hypothesis) there must be a path of odd length in $G - e$ between $u$ and $v$. Thus in the bipartite colouring of $G - e$ we have that $u$ and $v$ are different colours. Hence in both cases $G$ is bipartite.

2. Euler’s Theorem gives $v - e + f = 2$ so $8 - \frac{3 \times 4}{2} + f = 2$ and $f = 10$.

3. $Q_k$ is planar for $k \leq 3$ and $Q_k$ is not planar for $k \geq 4$.

For $k = 1, 2, 3$ we can easily draw $Q_k$ (do it) and see they are planar. For $k = 4$ note that since $Q_k$ contains no triangles we have that if $Q_4$ was planar then it would satisfy $e \leq 2v - 4$. However since $e = 32, v = 16$ using our formula from Q18 this is not satisfied and hence $Q_4$ is not planar.

For $k > 4$ consider a subgraph of $Q_k$ consisting of the set vertices whose last $k - 4$ digits are identical. Then this subgraph is isomorphic to $Q_4$ by the isomorphism $\phi : (a_1, a_2, a_3, a_4, \ldots) \mapsto (a_1, a_2, a_3, a_4)$, and so is not planar. Hence since a subgraph of $Q_k$ is not planar, then $Q_k$ is not planar.
4. Let the average degree of a vertex be denoted by $D = \frac{2e}{v}$. Then $e = \frac{Dv}{2}$. Substituting into $e \leq 3v - 6$ and rearranging we get

$$D \leq 6 - \frac{12}{v}.$$ 

Since $\frac{12}{v} > 0$ this implies that $D < 6$.

5. Since $G$ is a connected, planar, simple graph with $v$ vertices then if every face has $k$ edges, by Euler’s Theorem we have

$$v - \frac{fk}{2} + f = 2$$

$$\Rightarrow 2v - fk + 2f = 4$$

$$\Rightarrow f(k - 2) = 2v - 4$$

$$\Rightarrow f = \frac{2v - 4}{k - 2}.$$ 

Substituting back into Euler’s Theorem we get

$$e = -2 + f + v = -2 + \frac{2v - 4}{k - 2} + v = \frac{k(v - 2)}{k - 2}.$$