1. For the lower bound remove C and get a minimal spanning tree below of weight 6. Hence a lower bound is $6 + 4 + 4 = 14$.

For the upper bound a minimal spanning tree is as below. Starting at D we get DAEFBCD and an upper bound is 16.

```
lower bound
A -- B
|   |
|   |
F -- E -- D
```

```
upper bound
A -- B
|   |
|   |
F -- E -- D -- C
```

2. The statement is false for every $n \geq 1$.

For $n = 1$ we do not have 2 vertices so the claim is false.

For $n \geq 2$ consider the following orientation on $K_n$, which is a simple digraph. Label the vertices of $K_n$ by $v_1, \ldots, v_n$ and direct an edge from $v_i$ to $v_j$ if $i < j$. Then the indegree of $v_i$ is $i - 1$ and the out degree of $v_i$ is $(n - 1) - (i - 1) = n - i$. So no two vertices have the same indegree or outdegree.

3. Take a directed path of maximum length $v_1 \cdots v_k$, which we know exists since there are no directed cycles. Then $v_1$ must be a source otherwise we could add an incoming arc to $v_1$ to make a longer directed path. Similarly, $v_k$ must be a sink otherwise we could add an outgoing arc from $v_k$ to make a longer directed path.

For the second part we will do a strong induction on the number of vertices.

**Base case:** The result holds with a digraph consisting of one vertex.

**Induction step:** Assume the result is true for all digraphs without directed cycles and up to $n - 1$ vertices. Let $D$ be a digraph without directed cycles with $n$ vertices. By the first part we know $D$ contains a sink $v$. Delete $v$ to form $D'$ (that has no directed cycles as there were none in $D$) and by induction label the $n - 1$ vertices $1, \ldots, n - 1$ in $D'$. Reinsert $v$ and label it $n$. By construction the vertices of $D$ are ordered such that if an arc goes from vertex $i$ to vertex $j$ then $i < j$ for all $1 \leq i, j \leq n$, and the result follows by induction.
4. If \( n \) is even then the degree of every vertex is odd, and hence \( \text{indeg}(v) \neq \text{outdeg}(v) \) for any vertex \( v \). If \( n \) is odd, then the degree of every vertex is even. Let the vertices be \( v_1, \ldots, v_{2n-1} \). Then direct an edge between \( v_i \) and \( v_j \) where \( i > j \) from \( v_i \) to \( v_j \) if \( (i - j) \) is odd and from \( v_j \) to \( v_i \) if \( (i - j) \) is even. Then by construction \( \text{indeg}(v) = \text{outdeg}(v) \) for every vertex \( v \) in the graph.

5. Consider the partition with one vertex \( v \) in \( S \) and all the others in \( T \). Then there exists at least one arc from the vertex in \( S \) to a vertex in \( T \). Let \( V_1 \) be the set of all vertices in \( T \) at the end of such an arc. Now let \( S = \{ v \} \cup V_1 \), and all the others vertices be in \( T \), then there exists at least one arc from \( S \) to \( T \). Let \( V_2 \) be the set of all vertices in \( T \) at the end of such an arc. Now let \( S = \{ v \} \cup V_1 \cup V_2 \), and all the other vertices be in \( T \). Continuing in this way since our graph has a finite number of vertices we know this must terminate at some point.

If it terminates when \( S = V \) then we have by construction that there is a directed path from \( v \) to any other vertex and since \( v \) was chosen at random there is a directed path between any two vertices and by definition our graph is strongly connected.

If it terminates when \( S \neq V \) then there exists a partition of the vertices such that an arc does not go from any vertex in \( S \) to any vertex in \( T \) and so there is no directed path from \( v \) to any vertex in \( T \) and by definition our graph is not strongly connected.

6. Checking all cuts we find the minimum cut is 7, so a max flow is