1. Drawing a graph with a vertex for each lecture, and an edge between them if they must not coincide we get the following.

Notice there is a $K_4$ subgraph so we need at least 4 colours. If we colour $a$ and $e$ colour 1, $b$ and $f$ colour 2, $c$ and $g$ colour 3 and $d$ colour 4 we need at most 4 colours, so 4 periods are needed.

Alternatively, using deletion-contraction, we can compute the chromatic polynomial to be $k(k-1)^2(k-2)(k-3)(k^2-5k+8)$ again giving that 4 colours and hence periods are needed.

2. Draw a graph with a vertex for each team and an edge between them if they are playing a match. Now colour an edge $i$ if the match is being played in period $i$. This is equivalent to finding the chromatic index of $K_5$ which we know is 5 by Theorem 5. Hence 5 periods are needed and since 2 edges are coloured each colour and we have 2 pitches we are done.

3. If a simple connected graph is a tree then $f = 1$ and by Euler’s Theorem $v - e + f = 2 \Rightarrow v - e + 1 = 2 \Rightarrow v = e + 1$.
Conversely, if a simple connected graph has $v = e + 1$ then if we assume we have a cycle containing $m$ vertices it must contain $m$ edges. Consequently we have $v - m$ vertices to connect to the graph with $v - 1 - m$ edges, since a vertex needs at least one edge to connect it to the graph this is impossible, so there are no cycles, so we have a tree.

4. If we have a tree then since it is a simple connected graph there is exactly one path between any two vertices by Theorem 7 (so no cycles). Connecting two of them them creates exactly two paths between them, that is, one cycle, as since we had a tree there were none before.
Conversely, if adding one edge creates exactly one cycle in a simple connected graph then this means we started with a simple connected graph with no cycles, i.e. a tree.

5. Let our forest have $n$ vertices and $m$ components. Then by Q51 if component 1 has $v_1$ vertices then it has $v_1 - 1$ edges, ..., if component $m$ has $v_m$ vertices then it has $v_m - 1$ edges, and $v_1 + \cdots + v_m = n$. Thus

$$k = v_1 - 1 + \cdots + v_m - 1$$
$$= v_1 + \cdots + v_m - m$$
$$\Rightarrow m = v_1 + \cdots + v_m - k$$
$$= n - k$$

as desired.
6. We will do a proof by strong induction on the number of edges $E$.

*Base case:* If $E = 0$ then $T$ is a single vertex and $T$ has 1 leaf and 0 non-leaf vertices.

*Induction step:* Assume the result is true for all trees with fewer than $m - 1$ edges and consider a tree $T$ with $m - 1$ edges, $L$ leaves and $m - L$ non-leaves (since by Q51 $T$ has $m$ vertices in total). Delete a leaf $\ell$. If deleting $\ell$ results in no vertices of degree 2 then by induction

$$L - 1 > m - L \Rightarrow L > m - L.$$  

If deleting $\ell$ results in a vertex of degree 2, $v$, then this means that in $T$ that $v$ is adjacent to exactly 2 leaves, or $T = K_{1,3}$. If $T = K_{1,3}$ then the result follows. If not then delete both leaves $v$ is adjacent to, resulting in $v$ becoming a leaf. Then by induction

$$L - 2 + 1 > m - L - 1 \Rightarrow L > m - L.$$  

Hence the result follows by induction.