1. We will do a strong induction on the number of edges $E$.

Base case: $E = 1$. Let $G$ be a simple graph with $n$ vertices where $n \geq 2$ and one edge $e$. Then by deletion-contraction and Proposition 6 we have

$$P_G(k) = P_{N_n}(k) - P_{N_{n-1}}(k) = k^n - k^{n-1}$$

and the sum of the coefficients of $P_G(k)$ is 0.

Induction step: Now assume the result is true for a graph with $E$ edges where $1 < E < m$ and consider a simple graph $G$ with $m$ edges. Then by deletion-contraction

$$P_G(k) = P_{G-e}(k) - P_{G/e}(k)$$

and by the induction assumption the sum of the coefficients of $P_{G-e}(k)$ and $P_{G/e}(k)$ each sum to 0 and hence the coefficients of $P_G(k)$ sum to 0, and the result follows by induction.

2. At the point where we delete and reinsert a vertex of degree 5 in the five colour theorem we rely on the fact that $K_5$ is not planar. However, when we adjust the proof to delete and reinsert a vertex of degree 4 in proving the four colour theorem we would need that $K_4$ is not planar, but this is not true.

3. Let $\chi(G) = k$ and the deletion of any vertex $v$ yields a graph with a smaller chromatic number, i.e. $\chi(G - v) = m \leq k - 1$. If there exists a vertex $\tilde{v} \in V(G)$ of degree less than $k - 1$ delete it. Then colour the remaining graph in $k - 1$ colours. Reinsert $\tilde{v}$. As it is adjacent to at most $k - 2$ vertices there is at least one of the $k - 1$ colours we could colour it, so $\chi(G) \leq k - 1$. Since this is not true, the degree of every vertex must be at least $k - 1$.

4. is 3-colourable(f) and 3-colourable.

5. Recall that in $Q_k$ a vertex labelled by $a_1 \ldots a_k$ where $a_i = 0, 1$ for all $1 \leq i \leq k$ is adjacent to $k$ vertices since there are $k$ choices for where a sequence can differ from $a_1 \ldots a_k$ in just one place.

We can colour the edges of $Q_k$ with $k$ colours in the following way. If two adjacent vertices have their labels differing in position $i$ where $1 \leq i \leq k$ then colour the edge between them
colour \(i\). Note that by construction every edge incident at a vertex is thus a different colour, and hence we have an edge colouring of \(Q_k\) with \(k\) colours.

Consequently, \(\chi'(Q_k) \leq k\). However, since \(k\) edges meet at a vertex, \(\chi'(Q_k) \geq k\). Therefore \(\chi'(Q_k) = k\).

6. We have no loops as the graph is simple, so it is edge colourable. Assume it is edge colourable in \(d \geq 2\) colours for our \(d\) regular graph.

We know that since \(G\) is regular \(vd = 2e\), so \(v/2 = e/d\) so \(e/d\) is not an integer since \(v\) is odd. On the other hand since each vertex has \(d\) edges incident to it, coloured \(d\) colours, this means there are the same number of edges of each colour, say \(p\). So \(pd = e\) and \(e/d\) is an integer. We have a contradiction. Thus by Vizing’s Theorem, the chromatic index is \(d + 1\).