1. Each edge has 2 ends, and each end contributes 1 to the total sum of the degrees. Hence if there are \( e \) edges in the graph the total sum of the degrees is \( 2e \), which is even.

Let \( G \) be a graph with total sum of degrees \( 2e \). If we subtract the degrees of every vertex with even degree we are left with an even number of degrees \( 2t \). This must be the sum of the degrees of every vertex with odd degree. The only way this can happen is if the number of vertices of odd degree is even.

If we draw a graph with a vertex for each person at the party, and an edge between them if they have shaken hands, by the previous part of this question, if the vertex representing me has degree 5 then another vertex has odd degree, since the number of vertices of odd degree must be even and \( \geq 1 \).

2. If every vertex in \( G \) is adjacent to \( r \) of the other \( n - 1 \) vertices, then every vertex is not adjacent to \( n - r - 1 \) vertices. Therefore by the definition of \( \overline{G} \), every vertex in \( \overline{G} \) is adjacent to \( n - r - 1 \) vertices. Hence \( \overline{G} \) is regular of degree \( n - r - 1 \).

3. If an \( n \) vertex graph is self-complementary then we know its number of edges, \( e \), is half that in the complete graph \( K_n \), so \( e = \frac{1}{4} n(n - 1) \). However, \( e \) must be an integer and so \( n \) or \( n - 1 \) is a multiple of 4, or \( n = 4k \) or \( 4k + 1 \) for \( k \in \mathbb{Z} \).

4. We will do a proof by contradiction. Assume that every vertex \( v \) in \( G \) satisfies \( \deg(v) < 2n \). Then since \( G \) is self-complementary every vertex \( v \) in \( \overline{G} \) satisfies \( 4n - \deg(v) < 2n \), since the degree of \( v \) in \( \overline{G} \) is \( 4n - \deg(v) \) by definition of \( \overline{G} \). Hence

\[
4n = \deg(v) + 4n - \deg(v) < 2n + 2n = 4n
\]

a contradiction. Hence at least one vertex \( v \) in \( G \) satisfies \( \deg(v) \geq 2n \).

5. Labelling the graphs from left to right, top to bottom, looking at vertices we see: 3 of them (1, 2, 6) have 2 of degree 3, and 3 of degree 2; 1 (3) has 1 of degree 4, 1 of degree 3, 2 of degree 2, and 1 of degree 1; 1 (4) has 3 of degree 3, 1 of degree 2, and 1 of degree 1; 1 (5) has 1 of degree 4, and 4 of degree 2. Hence, although they all have 5 vertices and 6 edges, by degrees only 1, 2, 6 could be isomorphic. 1 and 6 are not isomorphic to 2 since the former have the 2 degree 3 vertices adjacent, and the latter doesn’t. An isomorphism between 1 and 6 is:

(\text{where } A \text{ maps to } a \text{ etc.})
6. The number of vertices of $Q_k$ is $2^k$

Proof By definition, each vertex corresponds to a sequence of $k$ symbols $(a_1, \ldots, a_k)$ where each $a_i = 0, 1$. So we have 2 choices for $a_1$, 2 choices for $a_2$, \ldots, 2 choices for $a_k$. That is $2^k$ choices in total, and hence $2^k$ vertices.

The number of edges of $Q_k$ is $k2^{k-1}$

Proof Consider a vertex $v$ corresponding to the sequence of 0s and 1s $(a_1, \ldots, a_k)$. Then $v$ is connected to $k$ other vertices each of which differ from $v$ in exactly one of $a_1, \ldots, a_k$. Now note that the total number of vertices by above is $2^k$, so the total number of edges coming out of some vertex is $k2^k$. However, we have now counted each edge twice, and hence the number of edges is $k2^k/2 = k2^{k-1}$.