Full Name: ______________________  Grade:
Student No: _____________________
1. (2 points)

(a) Find the 4th roots of $-8 + 8\sqrt{3}i$ and write them in the form $a + bi$.

**Solution:** Note that $-8 + 8\sqrt{3} = 16e^{\frac{2\pi}{3}}$. Therefore,

$$( -8 + 8\sqrt{3} )^{\frac{1}{4}} = \{ \sqrt[4]{16} e^{i\frac{2\pi}{3} + 2k\pi} : k \in \mathbb{Z} \}$$

$$= \{ 2e^{i\frac{2\pi}{3} + 2k\pi} : k \in \mathbb{Z} \}$$

$$= \{ 2e^{i\frac{2\pi}{3}}, 2e^{i\frac{4\pi}{3}}, 2e^{i\frac{6\pi}{3}}, 2e^{i\frac{8\pi}{3}} \}$$

$$= \{ \sqrt{3} + i, -1 + i\sqrt{3}, -\sqrt{3} - i, 1 - i\sqrt{3} \}$$

(b) Compute the integral $\int_{0}^{\pi} (\sin \theta)^2 \cos 2\theta \, d\theta$.

(HINT: Write $\sin$ and $\cos$ in terms of the exponential function and simplify.)

**Solution:** We have

$$(\sin \theta)^2 \cos 2\theta = \left( \frac{e^{i\theta} - e^{-i\theta}}{2i} \right)^2 \left( \frac{e^{i2\theta} + e^{-i2\theta}}{2} \right)$$

$$= -\frac{1}{8} (e^{i2\theta} + e^{-i2\theta} - 2)(e^{i2\theta} + e^{-i2\theta})$$

$$= -\frac{1}{8} (e^{i4\theta} + 1 + e^{-i4\theta} - 2e^{i2\theta} - 2e^{-i2\theta})$$

$$= -\frac{1}{4} \cos 4\theta + \frac{1}{2} \cos 2\theta - \frac{1}{4}.$$ 

Therefore,

$$\int_{0}^{\pi} (\sin \theta)^2 \cos 2\theta \, d\theta = \int_{0}^{\pi} \left[ -\frac{1}{4} \cos 4\theta + \frac{1}{2} \cos 2\theta - \frac{1}{4} \right] \, d\theta = -\frac{\pi}{4}.$$
2. (5 points) Let $A$ be the open disk of radius 2 centered at the origin, $B$ the set of points $z \in \mathbb{C}$ such that $|\text{Arg } z| > \pi/6$, and $C := \{z \in \mathbb{C} : \text{Re } z \geq 1\}$. Let $D$ be the intersection of $A$ and $B$, and let $E$ be the intersection of $C$ and $D$.

(a) Sketch the three sets $A$, $B$, and $C$ on the complex plane and clarify the boundary points.

Solution: Solid lines are included, dashed lines are excluded.

(b) Is $D$ a domain?
(A domain is a set that is both open and connected.)

Answer: Yes

(c) Is $E$ a region?
(A region is a domain together with some, none or all of its boundary points.)

Answer: No (not connected)
3. (3 points) Show (by means of a geometric or algebraic argument) that if \( u \) and \( v \) are two distinct points on the unit circle, then the real part of \( \frac{u + v}{u - v} \) must be zero.

**Solution:**

*Algebraic argument:*

Multiplying the numerator and the denominator by the conjugate of the denominator, we have

\[
\frac{u + v}{u - v} = \frac{u + v}{u - v} \cdot \frac{\overline{u} - \overline{v}}{\overline{u} - \overline{v}} = \frac{\overline{u}v - u\overline{v} + vv - vv}{|u - v|^2} = \frac{|u|^2 - |v|^2 + vv - u\overline{v}}{2|u - v|^2} = \frac{2i \text{Im}(v\overline{u})}{|u - v|^2},
\]

which is purely imaginary.

*Geometric argument:*

Call \( A, B, C \) the points represented by \( u, u + v \) and \( u - v \) respectively, and denote the origin by \( O \). The two triangles \( \triangle OAB \) and \( \triangle OAC \) are isosceles, because \( |OA| = |AB| = |AC| = 1 \). Therefore, \( \angle OBA = \angle AOB \) and \( \angle COA = \angle ACO \). Since the sum of the latter four angles is \( \pi \) (or \( -\pi \), depending on the orientation), we find that \( \angle COB = \angle COA + \angle AOB = \pi/2 \) (or \( -\pi/2 \), depending on the orientation). This means that \( u + v \) is perpendicular to \( u - v \). It follows that the principal argument of \( \frac{u + v}{u - v} \) is either \( \pi/2 \) or \( -\pi/2 \), which means its real part must be zero.