Metastability of the hard-core process on bipartite graphs

Frank den Hollander\textsuperscript{1}  Francesca Nardi\textsuperscript{2}  Siamak Taati\textsuperscript{1}

\textsuperscript{1}Mathematical Institute, Leiden University
\textsuperscript{2}Department of Mathematics, Eindhoven University of Technology

METASTABILITY Workshop
Eurandom, April 2016
Hard-core gas process

Configurations

- Each site can carry at most one particle.
- **Constraint**: particles cannot site next to each other.

Dynamics

- **Birth attempt** at site $k$ (Poisson clock with rate $\lambda_k$)
- **Death attempt** at site $k$ (Poisson clock with rate $1$)
- All clocks are independent.
Hard-core gas process

Configurations

- Each site can carry at most one particle.
- **Constraint**: particles cannot site next to each other.

Dynamics

- **Birth attempt** at site $k$ (Poisson clock with rate $\lambda_k$)
- **Death attempt** at site $k$ (Poisson clock with rate $1$)
- All clocks are independent.
Hard-core gas process

Configurations

- Each site can carry at most one particle.
- **Constraint**: particles cannot site next to each other.

Dynamics

- Birth attempt at site \( k \) (Poisson clock with rate \( \lambda_k \))
- Death attempt at site \( k \) (Poisson clock with rate 1)
- All clocks are independent.
Hard-core gas process

Configurations

- Each site can carry at most one particle.
- **Constraint**: particles cannot site next to each other.

Dynamics

- **Birth attempt** at site $k$ (Poisson clock with rate $\lambda_k$)
- **Death attempt** at site $k$ (Poisson clock with rate 1)
- All clocks are independent.
Hard-core gas process

Configurations

▶ Each site can carry at most one particle.
▶ Constraint: particles cannot site next to each other.

[Particles cannot overlap!]

Dynamics

▶ Birth attempt at site $k$ (Poisson clock with rate $\lambda_k$)
▶ Death attempt at site $k$ (Poisson clock with rate 1)
▶ All clocks are independent.
Hard-core gas process

Configurations

- Each site can carry at most one particle.
- **Constraint:** particles cannot site next to each other.

Dynamics

- **Birth attempt** at site $k$ (Poisson clock with rate $\lambda_k$)
- **Death attempt** at site $k$ (Poisson clock with rate 1)
- All clocks are independent.
Hard-core gas process

Configurations

- Each site can carry at most one particle.
- **Constraint:** particles cannot site next to each other.

Dynamics

- **Birth attempt** at site $k$ (Poisson clock with rate $\lambda_k$)
- **Death attempt** at site $k$ (Poisson clock with rate $1$)
- All clocks are independent.
Hard-core gas process

Reversible stationary distribution

\[ \pi(x) = \frac{1}{Z} \prod_{k \text{ occupied in } x} \lambda_k \]

for each valid configuration \( x \).

(\( Z \) is the appropriate normalizing constant.)
Hard-core gas process

Asymptotic regime
When the birth rates $\lambda_k$ are very large:

\[ \pi(x) = \frac{1}{Z} \prod_{k \text{ occupied in } x} \lambda_k \]
Hard-core gas process

Asymptotic regime
When the birth rates $\lambda_k$ are very large:

- The process tends to remain close to fully packed configurations, specially those that are “locally stable”.

$$\pi(x) = \frac{1}{Z} \prod_{k \text{ occupied in } x} \lambda_k$$
Hard-core gas process

Asymptotic regime
When the birth rates $\lambda_k$ are very large:

- The process tends to remain close to **fully packed** configurations, specially those that are “locally stable”.

Reversible stationary distribution

$$\pi(x) = \frac{1}{Z} \prod_{k \text{ occupied in } x} \lambda_k$$
Hard-core gas process

Asymptotic regime
When the birth rates $\lambda_k$ are very large:

▶ The process tends to remain close to fully packed configurations, specially those that are “locally stable”.

\[
\pi(x) = \frac{1}{Z} \prod_{k \text{ occupied in } x} \lambda_k
\]
Hard-core gas process

Asymptotic regime
When the birth rates $\lambda_k$ are very large:

- The process tends to remain close to fully packed configurations, specially those that are “locally stable”.

$$\pi(x) = \frac{1}{Z} \prod_{k \text{ occupied in } x} \lambda_k$$

fully packed but not “stable”

reversible stationary distribution

$\pi(x) = \frac{1}{Z} \prod_{k \text{ occupied in } x} \lambda_k$
Hard-core gas process

Asymptotic regime
When the birth rates $\lambda_k$ are very large:

- The process tends to remain close to fully packed configurations, specially those that are "locally stable".

$$\pi(x) = \frac{1}{Z} \prod_{k \text{ occupied in } x} \lambda_k$$
Hard-core gas process

Asymptotic regime
When the birth rates $\lambda_k$ are very large:

- The process tends to remain close to fully packed configurations, specially those that are “locally stable”.

Reversible stationary distribution

$$\pi(x) = \frac{1}{Z} \prod_{k \text{ occupied in } x} \lambda_k$$
Hard-core gas process

Asymptotic regime

When the birth rates $\lambda_k$ are very large:

- The process tends to remain close to fully packed configurations, specially those that are “locally stable”.
- A typical stationary sample is efficiently fully packed!
Hard-core gas process

Fully packed and “stable” and efficient (12)

Asymptotic regime
When the birth rates $\lambda_k$ are very large:

- The process tends to remain close to fully packed configurations, specially those that are “locally stable”.
- A typical stationary sample is efficiently fully packed!
Hard-core gas process

Metastability

- It takes a long time for the process to leave a “locally stable” but inefficiently packed configuration. [large exit time]
- Once a more efficient configuration is reached, it takes much longer to return. [small stationary probability]
Hard-core gas process

Asymptotic regime
birth rates $\lambda_k$ large

Stationary distribution
$$\pi(x) = \frac{1}{Z} \prod_{k \text{ occupied in } x} \lambda_k$$

Metastability

- It takes a long time for the process to leave a “locally stable” but inefficiently packed configuration. [large exit time]
- Once a more efficient configuration is reached, it takes much longer to return. [small stationary probability]
Hard-core gas process

Metastability

- It takes a long time for the process to leave a “locally stable” but inefficiently packed configuration. [large exit time]
- Once a more efficient configuration is reached, it takes much longer to return. [small stationary probability]
Hard-core gas on graphs

Motivation

▸ classic example from statistical mechanics [on the lattice]
  ➡ phase transition (solid-gas) with symmetry breaking

▸ wireless communication networks
  ➡ the graph represents the possibilities of interference
  ➡ metastability undermines the network performance

▸ includes the Widom-Rowlinson model

Related work

▸ Galvin and Tetali (2006), Randall (2008),
  — and Antonio Blanca (2012)
Hard-core gas process

An exagerated example
- complete bipartite graph
- birth rate $\lambda$ at each site
- $\lambda$ large
- $|U| < |V|$

Intuitive observations
- Exactly two fully packed configurations $u$ and $v$
  - Both $u$ and $v$ are “locally stable”.
  - $v$ is “more efficient” than $u$.
- Metastable behaviour starting from $u$
Hard-core gas process

An exagerated example

- complete bipartite graph
- birth rate $\lambda$ at each site
- $\lambda$ large
- $|U| < |V|$ 

Intuitive observations

- Exactly two fully packed configurations $u$ and $v$
  - $\rightarrow$ Both $u$ and $v$ are “locally stable”.
  - $\rightarrow$ $v$ is “more efficient” than $u$.
- Metastable behaviour starting from $u$
Hard-core gas process

An exagerated example

- complete bipartite graph
- birth rate $\lambda$ at each site
- $\lambda$ large
- $|U| < |V|$

Intuitive observations

- Exactly two fully packed configurations $u$ and $v$
  - Both $u$ and $v$ are “locally stable”.
  - $v$ is “more efficient” than $u$.
- Metastable behaviour starting from $u$
Hard-core gas process

An exagerated example

- complete bipartite graph
- birth rate $\lambda$ at each site
- $\lambda$ large
- $|U| < |V|

Intuitive observations

- Exactly two fully packed configurations $u$ and $v$
  - $\rightarrow$ Both $u$ and $v$ are “locally stable”.
  - $\rightarrow$ $v$ is “more efficient” than $u$.
- Metastable behaviour starting from $u$
Hard-core gas process

An exagerated example

- complete bipartite graph
- birth rate $\lambda$ at each site
- $\lambda$ large
- $|U| < |V|$

Intuitive observations

- Exactly two fully packed configurations $u$ and $v$
  - $u$ and $v$ are “locally stable”.
  - $v$ is “more efficient” than $u$.
- Metastable behaviour starting from $u$
Hard-core gas process

An exagerated example

- complete bipartite graph
- birth rate $\lambda$ at each site
- $\lambda$ large
- $|U| < |V|

Intuitive observations

- Exactly two fully packed configurations $u$ and $v$
  - Both $u$ and $v$ are “locally stable”.
  - $v$ is “more efficient” than $u$.
- Metastable behaviour starting from $u$
Hard-core gas process

An exagerated example

- complete bipartite graph
- birth rate $\lambda$ at each site
- $\lambda$ large
- $|U| < |V|

Intuitive observations

- Exactly two fully packed configurations $u$ and $v$
  - $u$ and $v$ are “locally stable”.
  - $v$ is “more efficient” than $u$.
- Metastable behaviour starting from $u$
Hard-core gas process

An exaggerated example

- complete bipartite graph
- birth rate $\lambda$ at each site
- $\lambda$ large
- $|U| < |V|$

Intuitive observations

- Exactly two fully packed configurations $u$ and $v$
  - $u$ and $v$ are “locally stable”.
  - $v$ is “more efficient” than $u$.
- Metastable behaviour starting from $u$
Hard-core gas process

An exagerated example

- complete bipartite graph
- birth rate $\lambda$ at each site
- $\lambda$ large
- $|U| < |V|

Intuitive observations

- Exactly two fully packed configurations $u$ and $v$
  - Both $u$ and $v$ are “locally stable”.
  - $v$ is “more efficient” than $u$.
- Metastable behaviour starting from $u$
Hard-core gas process

An exagerated example

- complete bipartite graph
- birth rate $\lambda$ at each site
- $\lambda$ large
- $|U| < |V|

Intuitive observations

- Exactly two fully packed configurations $u$ and $v$
  - $\longrightarrow$ Both $u$ and $v$ are “locally stable”.
  - $\longrightarrow$ $v$ is “more efficient” than $u$.
- Metastable behaviour starting from $u$
Hard-core gas process

An exagerated example

- complete bipartite graph
- birth rate $\lambda$ at each site
- $\lambda$ large
- $|U| < |V|$ 

Intuitive observations

- Exactly two fully packed configurations $u$ and $v$
  - $\rightarrow$ Both $u$ and $v$ are “locally stable”.
  - $\rightarrow$ $v$ is “more efficient” than $u$.
- Metastable behaviour starting from $u$
Hard-core gas process

An exagerated example

- complete bipartite graph
- birth rate $\lambda$ at each site
- $\lambda$ large
- $|U| < |V|$

Intuitive observations

- Exactly two fully packed configurations $u$ and $v$
  - Both $u$ and $v$ are “locally stable”.
  - $v$ is “more efficient” than $u$.
- Metastable behaviour starting from $u$
Hard-core gas process

An exaggerated example

- complete bipartite graph
- birth rate $\lambda$ at each site
- $\lambda$ large
- $|U| < |V|$

Intuitive observations

- Exactly two fully packed configurations $u$ and $v$
  - Both $u$ and $v$ are “locally stable”.
  - $v$ is “more efficient” than $u$.
- Metastable behaviour starting from $u$
Hard-core gas process

An exaggerated example

- complete bipartite graph
- birth rate $\lambda$ at each site
- $\lambda$ large
- $|U| < |V|

Intuitive observations

- Exactly two fully packed configurations $u$ and $v$
- Both $u$ and $v$ are “locally stable”.
- $v$ is “more efficient” than $u$.
- Metastable behaviour starting from $u$
Hard-core gas process

An exagerated example

- complete bipartite graph
- birth rate $\lambda$ at each site
- $\lambda$ large
- $|U| < |V|$

Intuitive observations

- Exactly two fully packed configurations $u$ and $v$
  - $u$ and $v$ are “locally stable”.
  - $v$ is “more efficient” than $u$.
- Metastable behaviour starting from $u$
Hard-core gas process

An exaggerated example

- complete bipartite graph
- birth rate $\lambda$ at each site
- $\lambda$ large
- $|U| < |V|$

Intuitive observations

- Exactly two fully packed configurations $u$ and $v$
  - $u$ and $v$ are “locally stable”.
  - $v$ is “more efficient” than $u$.
- Metastable behaviour starting from $u$
Hard-core gas process

An exagerated example

- complete bipartite graph
- birth rate $\lambda$ at each site
- $\lambda$ large
- $|U| < |V|$

Intuitive observations

- Exactly two fully packed configurations $u$ and $v$
  - $u$ and $v$ are “locally stable”.
  - $v$ is “more efficient” than $u$.
- Metastable behaviour starting from $u$
Hard-core gas process

An exagerated example

- complete bipartite graph
- birth rate $\lambda$ at each site
- $\lambda$ large
- $|U| < |V|

Intuitive observations

- Exactly two fully packed configurations $u$ and $v$
  - $u$ and $v$ are “locally stable”.
  - $v$ is “more efficient” than $u$.
- Metastable behaviour starting from $u$
Hard-core gas process

An exagerated example

- complete bipartite graph
- birth rate $\lambda$ at each site
- $\lambda$ large
- $|U| < |V|

Intuitive observations

- Exactly two fully packed configurations $u$ and $v$
  - Both $u$ and $v$ are “locally stable”.
  - $v$ is “more efficient” than $u$.
- Metastable behaviour starting from $u$

Question

*How long does the transition from $u$ to $v$ take?*
Hard-core process on a complete bipartite graph

reversible Markov chain
Let $X_n (n \geq 0)$ be the discrete-time Markov chain. The first hitting time of $v$ is

$$T_v := \inf\{n \geq 0 : X_n = v\}.$$
Let $X_n\ (n \geq 0)$ be the discrete-time Markov chain. The first hitting time of $v$ is

$$T_v := \inf\{n \geq 0 : X_n = v\}.$$

Let $X_n \ (n \geq 0)$ be the discrete-time Markov chain. The first hitting time of $v$ is

$$T_v := \inf\{n \geq 0 : X_n = v\}.$$ 

**Question**

*What is the expected transition time $\mathbb{E}_{u} T_v$?*
Hard-core process on a complete bipartite graph

As an electric network

![Diagram of a complete bipartite graph with electric network representation]
Review: reversible Markov chain vs. electric network

Fundamental connection I
For every state $x$,

$$P_x(T_A < T_B) = \text{voltage}(x)$$

if a $1^V$ battery is connected between $A$ and $B$.

Fundamental connection II
For every state $x$,

$$G_{TB}(a, x) = R(a \leftrightarrow B) \pi(x) P_x(T_a < T_B)$$

where $G_{TB}(a, x) := \mathbb{E}_a[\# \text{ of visits to } x \text{ before } T_B]$.

Corollary

$$\mathbb{E}_a T_B = R(a \leftrightarrow B) \sum_x \pi(x) P_x(T_a < T_B)$$
$E_u T_v \approx E_u T_Z$

$$= \mathcal{R}(u \leftrightarrow Z) \sum_x \pi(x) \mathbb{P}_x (T_u < T_Z)$$

$$= \pi(u) \mathcal{R}(u \leftrightarrow Z) \sum_x \frac{\pi(x)}{\pi(u)} \mathbb{P}_x (T_u < T_Z)$$

$$\approx \pi(u) \mathcal{R}(u \leftrightarrow Z)$$
Hard-core process on a complete bipartite graph

Expected transition time

\[ \mathbb{E}_u T_v \approx \mathbb{E}_u T_Z \]

\[ = \mathcal{R}(u \leftrightarrow Z) \sum_x \pi(x) \mathbb{P}_x(T_u < T_Z) \]

\[ = \pi(u) \mathcal{R}(u \leftrightarrow Z) \sum_x \frac{\pi(x)}{\pi(u)} \mathbb{P}_x(T_u < T_Z) \]

\[ \approx \pi(u) \mathcal{R}(u \leftrightarrow Z) \]
Hard-core process on a complete bipartite graph

Expected transition time

\[
\mathbb{E}_u T_v \approx \mathbb{E}_u T_Z = R(u \leftrightarrow Z) \sum_x \pi(x) P_x(T_u < T_Z)
\]

\[
= \pi(u) R(u \leftrightarrow Z) \sum_x \frac{\pi(x)}{\pi(u)} P_x(T_u < T_Z)
\]

\[
\approx \pi(u) R(u \leftrightarrow Z)
\]
Hard-core process on a complete bipartite graph

Expected transition time

\[ E_u T_v \approx E_u T_Z \]

\[ = R(u \leftrightarrow Z) \sum_x \pi(x) P_x(T_u < T_Z) \]

\[ = \pi(u)R(u \leftrightarrow Z) \left[ 1 + \sum_{x \neq u} \frac{\pi(x)}{\pi(u)} P_x(T_u < T_Z) \right] \]

\[ \approx \pi(u)R(u \leftrightarrow Z) \]
Hard-core process on a complete bipartite graph

Expected transition time

\[
\mathbb{E}_u T_v \approx \mathbb{E}_u T_Z
\]

\[
= \mathcal{R}(u \leftrightarrow Z) \sum_x \pi(x) \mathbb{P}_x(T_u < T_Z)
\]

\[
= \pi(u) \mathcal{R}(u \leftrightarrow Z) \left[ 1 + \sum_{x \neq u} \frac{\pi(x)}{\pi(u)} \mathbb{P}_x(T_u < T_Z) \right]
\]

\[
\approx \pi(u) \mathcal{R}(u \leftrightarrow Z)
\]
Hard-core process on a complete bipartite graph

Expected transition time

\[ E_u T_v \approx \pi(u) \mathcal{R}(u \leftrightarrow Z) \approx \pi(u) \mathcal{R}(u \leftrightarrow v) \]
Hard-core process on a complete bipartite graph

Expected transition time

\[ E_u T_v \approx \pi(u)R(u \leftrightarrow Z) \approx \pi(u)R(u \leftrightarrow v) \]

It remains to estimate \( R(u \leftrightarrow v) \).
Hard-core process on a complete bipartite graph

Estimating the effective resistance
Hard-core process on a complete bipartite graph

Estimating the effective resistance
Hard-core process on a complete bipartite graph

Estimating the effective resistance
Hard-core process on a complete bipartite graph

Expected transition time

Proposition (Discrete time)

$$\mathbb{E}_u T_v = \frac{1}{|U|} \lambda^{|U|-1} [1 + o(1)]$$  as $\lambda \to \infty$. 
Hard-core process on a complete bipartite graph

Expected transition time

Proposition (Continuous time)

\[ \mathbb{E}_u T_v = \frac{\gamma}{|U|} \lambda^{|U|-1}[1 + o(1)] \text{ as } \lambda \to \infty. \]

\[ \gamma := (|U| + |V|)(1 + \lambda) \text{ is the rate of Poisson clock} \]
Hard-core process on a bipartite graph

A more general setting

- an arbitrary bipartite graph
- Birth rates
  \[ \lambda \quad \text{on } U \]
  \[ \bar{\lambda} \quad \text{on } V \]
- \( \lambda, \bar{\lambda} \) large

\[ \text{Intuitive observations} \]
- Two fully packed configurations \( u \) and \( v \) [but possibly many more]
- \( u \) and \( v \) are "locally stable".
- \( v \) is the "most efficient" packing.
Hard-core process on a bipartite graph

A more general setting

- an arbitrary bipartite graph
- Birth rates
  \[ \lambda \quad \text{on} \quad U \]
  \[ \bar{\lambda} \quad \text{on} \quad V \]
- \( \lambda, \bar{\lambda} \) large

Intuitive observations

- Two fully packed configurations \( u \) and \( v \) [but possibly many more]
  \[ \rightarrow \]
  - Both \( u \) and \( v \) are "locally stable".
  - \( v \) is the "most efficient" packing.
Hard-core process on a bipartite graph

A more general setting

- an arbitrary bipartite graph
- Birth rates
  \[ \lambda \quad \text{on} \quad U \]
  \[ \bar{\lambda} \quad \text{on} \quad V \]
- \( \lambda, \bar{\lambda} \) large
Hard-core process on a bipartite graph

A more general setting

- an arbitrary bipartite graph
- Birth rates (with $0 < \alpha < 1$)
  \[
  \lambda \quad \text{on } U \\
  \lambda = \lambda^{1+\alpha+o(1)} \quad \text{on } V
  \]
- $\lambda, \lambda$ large

Intuitive observations

- Two fully packed configurations $u$ and $v$ [but possibly many more]
- Both $u$ and $v$ are "locally stable".
- $v$ is the "most efficient" packing.
Hard-core process on a bipartite graph

A more general setting

- an arbitrary bipartite graph
- Birth rates (with $0 < \alpha < 1$)
  \begin{align*}
  \lambda & \quad \text{on } U \\
  \bar{\lambda} & = \lambda^{1+\alpha+o(1)} \quad \text{on } V
  \end{align*}
- $\lambda, \bar{\lambda}$ large
- $|U| < (1 + \alpha) |V|$
Hard-core process on a bipartite graph

A more general setting

- an arbitrary bipartite graph
- Birth rates (with $0 < \alpha < 1$)
  \[ \lambda \quad \text{on } U \]
  \[ \bar{\lambda} = \lambda^{1+\alpha+o(1)} \quad \text{on } V \]
- $\lambda, \bar{\lambda}$ large
- $|U| < (1 + \alpha) |V|$

Intuitive observations

- Two fully packed configurations $u$ and $v$ [but possibly many more]
  - Both $u$ and $v$ are “locally stable”.
  - $v$ is the “most efficient” packing.
Hard-core process on a bipartite graph

A more general setting

- an arbitrary bipartite graph
- Birth rates (with $0 < \alpha < 1$)
  \[
  \lambda \quad \text{on } U \\
  \bar{\lambda} = \lambda^{1+\alpha+o(1)} \quad \text{on } V
  \]
- $\lambda, \bar{\lambda}$ large
- $|U| < (1 + \alpha) |V|$

Intuitive observations

- Two fully packed configurations $u$ and $v$ [but possibly many more]
  - Both $u$ and $v$ are “locally stable”.
  - $v$ is the “most efficient” packing.
Hard-core process on a bipartite graph

A more general setting

- an arbitrary bipartite graph
- Birth rates (with $0 < \alpha < 1$)
  \[
  \lambda \quad \text{on} \quad U
  \]
  \[
  \bar{\lambda} = \lambda^{1+\alpha+o(1)} \quad \text{on} \quad V
  \]
- $\lambda, \bar{\lambda}$ large
- $|U| < (1 + \alpha)|V|

Intuitive observations

- Two fully packed configurations $u$ and $v$ [but possibly many more]
  - Both $u$ and $v$ are “locally stable”.
  - $v$ is the “most efficient” packing.
Hard-core process on a bipartite graph

A more general setting

- an arbitrary bipartite graph
- Birth rates (with $0 < \alpha < 1$)
  \[ \lambda \text{ on } U \]
  \[ \bar{\lambda} = \lambda^{1+\alpha+o(1)} \text{ on } V \]
- $\lambda, \bar{\lambda}$ large
- $|U| < (1 + \alpha) |V|$

Intuitive observations

- Two fully packed configurations $u$ and $v$ [but possibly many more]
  - Both $u$ and $v$ are “locally stable”.
  - $v$ is the “most efficient” packing.
Hard-core process on a bipartite graph

Examples of bipartite graphs

graphs arising from two-species
Widom-Rowlinson model
Metastability in Markov processes

Some references

- Kramers (1940)
- large deviations / path-wise approach
  - Freidlin and Wentzell (1960–1970)
  - Cassandro, Galves, Olivieri and Vares (1984–)
  - .

- potential-theoretic approach
  - Bovier, Eckhoff, Gayrard and Klein (2001–)
  - .

Three books

- Olivieri and Vares: *Large Deviations and Metastability* (2005)
Main results: I

Theorem (Critical droplets)

For the hard-core dynamics on an even torus \( \mathbb{Z}_m \times \mathbb{Z}_n \), when going from \( u \) to \( v \), with large probability, the chain passes through exactly one transition \( Q \rightarrow Q^* \), where \( Q \) and \( Q^* \) are obtained from the solutions of an isoperimetric problem.

A configuration in \( Q \) [similar for hypercube]

[similar for Widom-Rowlinson]
Main results: I

Theorem (Critical droplets)

For the hard-core dynamics on an even torus $\mathbb{Z}_m \times \mathbb{Z}_n$, when going from $u$ to $v$, with large probability, the chain passes through exactly one transition $Q \to Q^*$, where $Q$ and $Q^*$ are obtained from the solutions of an isoperimetric problem.

A configuration in $Q^*$

[similar for hypercube]

[similar for Widom-Rowlinson]
Main results: II

Theorem (Expected transition time)

For the hard-core dynamics on an even torus $\mathbb{Z}_m \times \mathbb{Z}_n$ we have

$$\mathbb{E}_u T_v = \frac{\gamma}{2 m n l^*} \frac{\lambda^{l^*(l^*+1)+1}}{\lambda^{l^*(l^*-1)}} [1 + o(1)]$$

as $\lambda \to \infty$, where $l^* := \lceil \frac{1}{\alpha} \rceil$ is the size of the critical droplet and $\gamma := |U|(1 + \lambda) + |V|(1 + \bar{\lambda})$ is the rate of the Poisson clock.

[Similar for hypercube]

[Similar for Widom-Rowlinson]

Proof steps.

Show that (in discrete time)

$$\mathbb{E}_u T_v = \pi(u) \mathcal{R}(u \leftrightarrow v) [1 + o(1)]$$

as $\lambda \to \infty$.

Estimate the effective resistance.
Main results: III

Theorem (Asymptotic exponential law)

For the hard-core dynamics on “many” bipartite graphs we have

\[ P_u \left( \frac{T_v}{E_u T_v} > t \right) \rightarrow e^{-t} \]

uniformly in \( t \in \mathbb{R}^+ \) as \( \lambda \rightarrow \infty \).

Intuition.

Many many trials (attempts to form a critical droplet) with tiny probability of success

\[ \implies \text{ success time approximately exponential} \]
Effective resistance: rough estimate

Critical resistance
For every two states \( a, b \in \mathcal{X} \), set

\[
\Psi(a, b) := \inf_{\omega : a \leftrightarrow b} \sup_{e \in \omega} r(e)
\]

Remark

\( a, b \mapsto R(a \leftrightarrow b) \) is a metric on \( \mathcal{X} \).

\( a, b \mapsto \Psi(a, b) \) is an ultra-metric on \( \mathcal{X} \).

Proposition (Equivalence)

There exists a constant \( k \geq 1 \) such that \([\text{independent of } \lambda]\)

\[
\frac{1}{k} \Psi(a, b) \leq R(a \leftrightarrow b) \leq k \Psi(a, b)
\]

for all \( a, b \in \mathcal{X} \).
Effective resistance: sharp estimate

A pair \((Q, Q^*)\) is a critical gate between \(A\) and \(B\) if

1. \(r(x, y) \simeq \Psi(A, B)\) for every \(x \in Q\) and \(y \in Q^*\) with \(x \sim y\),
2. \(\Psi(A, x) \prec \Psi(A, B)\) for every \(x \in Q\),
3. \(\Psi(y, B) \prec \Psi(A, B)\) for every \(y \in Q^*\), and
4. every optimal path from \(A\) to \(B\) passes through a transition \(Q \rightarrow Q^*\).
Effective resistance: sharp estimate

Critical gate

Proposition

Let \((Q, Q^*)\) be a critical pair between \(A\) and \(B\). Then,

\[
C(A \leftrightarrow B) = c(Q, Q^*) \left[1 + o(1)\right]
\]

as \(\lambda \to \infty\),

where \(c(Q, Q^*) := \sum_{x \in Q} \sum_{y \in Q^*} c(x, y).\)
Effective resistance: sharp estimate

Critical gate

\[ C(A \leftrightarrow B) = c(Q, Q^*) [1 + o(1)] \]

Proof.

Upper bound: simple Nash-Williams inequality

Lower bound: generalized Nash-Williams inequality

[a.k.a. Berman-Konsowa variational principle]
Thank you for your attention!