1. (a) Let $Q(t)$ be the amount of salt (in g) in the tank at time $t$ in min. Salt enters that tank at a rate of

$$\text{rate in} = 2 \cdot \frac{1}{4} \left(1 + \frac{1}{2} \sin t\right) \text{ g/min},$$

and leaves at a rate of

$$\text{rate out} = 2 \cdot \frac{Q}{100} \text{ g/min},$$

so the differential equation for the rate at which the amount of salt in the tank changes is

$$\frac{dQ}{dt} = \text{rate in} - \text{rate out} = \frac{1}{2} + \frac{1}{4} \sin t - \frac{1}{50}Q,$$

(the units of this equation are g/min), and the initial condition is

$$Q(0) = 50.$$

The equation is linear. Write it as

$$Q' + \frac{1}{50}Q = \frac{1}{2} + \frac{1}{4} \sin t,$$

multiply by the integrating factor

$$\mu(t) = e^{t/50},$$

and get

$$(e^{t/50}Q)' = \frac{1}{2}e^{t/50} + \frac{1}{4}e^{t/50} \sin t.$$

Integrating both sides of the equation (the second term on the right-hand side requires integration by parts, twice) gives

$$e^{t/50}Q = 25 e^{t/50} + \frac{25}{5002} e^{t/50} \sin t - \frac{625}{2501} e^{t/50} \cos t + c,$$

$$Q = 25 + \frac{25}{5002} \sin t - \frac{625}{2501} \cos t + c e^{-t/50},$$

where $c$ is an arbitrary constant.

Using the initial condition we get

$$50 = 25 - \frac{625}{2501} + c,$$

$$c = \frac{63150}{2501},$$

so the amount of salt in the tank at any time $t$ is

$$Q(t) = 25 + \frac{25}{5002} \sin t - \frac{625}{2501} \cos t + \frac{63150}{2501} e^{-t/50} \text{ grams.}$$

(b) After a very long time the exponential term is negligible and

$$Q(t) \approx 25 + \frac{25}{5002} \sin t - \frac{625}{2501} \cos t,$$
which is an oscillation about the constant level

\[ 25 \text{ g} , \]

with amplitude

\[ \sqrt{\left( \frac{25}{5002} \right)^2 + \left( -\frac{625}{5001} \right)^2} = \frac{25\sqrt{2501}}{5002} \approx 0.25 \text{ g}. \]

2. First note that the volume \( V \) of brine is not constant. Its rate of change with respect to time \( t \) is

\[ \frac{dV}{dt} = \text{rate in} - \text{rate out} \]
\[ = 3 - 2 \text{ L/min} \]
\[ = 1 \text{ L/min} \]

and the initial volume is given as

\[ V(0) = 300 \text{ L}, \]

so the volume at time \( t \) min is

\[ V(t) = t + 300 \text{ L}. \]

(This does not have to be “derived” so explicitly; the volume formula for \( V(t) \) is “obvious” enough that it could be written down without explanation.) The tank begins to overflow the instant that \( V(t) = 1000 \text{ L} \), at

\[ t = 700 \text{ min}. \]

Let \( Q \) be the amount of salt in the tank. Its rate of change is

\[ \frac{dQ}{dt} = \text{rate in} - \text{rate out} \]
\[ = 3 \cdot 2 - 2 \cdot \frac{Q}{V} \text{ g/min} \]
\[ = 6 - \frac{2}{t + 300} \text{ g/min} \]

and the initial amount is given as

\[ Q(0) = 50 \text{ g}. \]

Write the differential equation in standard form as

\[ Q' + \frac{2}{t+300} Q = 6, \]

multiply both sides by the integrating factor

\[ \mu(t) = e^\int \frac{2}{t+300} \, dt = e^{2 \ln(t+300)} = (t + 300)^2 \]

to obtain

\[ (t + 300)^2 Q' + 2(t + 300)Q = 6(t + 300)^2, \]
rewrite the left-hand side as the derivative of a product

\[ [(t + 300)^2 Q)' = 6(t + 300)^2, \]

and integrate both sides to get

\[ (t + 300)^2 Q = \int 6(t + 300)^2 \, dt = 2(t + 300)^3 + c, \]

where \( c \) is an arbitrary constant. Then

\[ Q = 2(t + 300) + \frac{c}{(t + 300)^2}. \]

The initial condition \( Q = 50 \) when \( t = 0 \) gives

\[ c = -550 (300)^2, \]

and the amount of salt in the tank at any time \( t \) in min (until the tank overflows) is

\[ Q(t) = 2(t + 300) - 550 \left( \frac{300}{t + 300} \right)^2 \text{ g}. \]

At the instant the tank begins to overflow at \( t = 700 \) min, the amount of salt in the tank is

\[ Q(700) = 2(1000) - 550 \left( \frac{300}{1000} \right)^2 (= 1950.5) \text{ g}. \]

3. Acceleration is

\[ a = \frac{dv}{dt} = -\gamma v^2. \]

(a) Substituting \( dv/dt = v(dv/dx) \) into the above equation and simplifying, we get

\[ \frac{dv}{dx} = -\gamma v. \]

(b) The solution is

\[ v(x) = v_0 e^{-\gamma x}, \]

where \( v_0 \) is the velocity at \( x = 0 \);

\[ v_0 = 150 \text{ km/hr}, \]

and we can determine \( \gamma \) by setting \( v = 15 \text{ km/hr} \) when \( x = 500 \text{ m} \). We get

\[ 15 = 150e^{-0.5\gamma}, \]

and solving for \( \gamma \) gives

\[ \gamma = -2 \ln \left( \frac{15}{150} \right) = 2 \ln 10 \approx 4.605 \text{ km}^{-1}. \]
To find how long it takes, we need the original differential equation

\[ \frac{dv}{dt} = -\gamma v^2, \]

because this involves time \( t \). This last equation is separable, and separating variables, integrating and finding the constant of integration (by substituting \( t = 0, \ v = v_0 \)) gives

\[ -\int v^{-2} dv = \gamma \int dt \]

\[ \frac{1}{v} = \gamma t + c \]

\[ \frac{1}{v} = \gamma t + \frac{1}{v_0}. \]

Now we solve for \( t \),

\[ t = \frac{1}{\gamma} \left( \frac{1}{v} - \frac{1}{v_0} \right) = \frac{1}{2\ln 10} \left( \frac{1}{15} - \frac{1}{150} \right) (\approx 0.01303) \text{ hr} \]

or approximately 46.9 sec.

4. The equation is separable. Write it as

\[ y^{-1/2} dy = dt \]

and integrate to get

\[ 2y^{1/2} = t + c, \]

where \( c \) is an arbitrary constant. Then the initial condition \( y(0) = 0 \) gives \( c = 0 \), so

\[ 2y^{1/2} = t \quad (\text{notice this implies } t \geq 0), \]

\[ y = \phi(t) = \frac{1}{4} t^2, \]

is a solution (for \( t \geq 0 \)) as can be verified by direct substitution into the differential equation and initial condition. Another solution can be found by inspection (“guess and check”):

\[ y = \phi(t) = 0 \quad \text{for all } -\infty < t < \infty. \]

(Verify it is a solution by direct substitution into the differential equation and initial condition.) (In fact, you can verify by direct substitution that the continuous function

\[ y = \phi(t) = \begin{cases} 0 & \text{if } -\infty < t < t_0 \\ \frac{1}{2}(t - t_0)^2 & \text{if } t_0 \leq t < \infty \end{cases} \]

is a solution for any fixed \( t_0 \geq 0 \). These functions are all continuous with intervals of definition \( -\infty < t < \infty \).

Obviously Theorem 2.4.2 does not apply, because the conclusions do not hold (there is no unique solution to the initial value problem). This must mean at least one of the hypotheses is violated. To see which one, look at the continuity of \( f(t, y) \) and \( \partial f / \partial y(t, y) \) in the \( ty \)-plane.

\[ f(t, y) = \sqrt{y} \]
is continuous for all \( t \) and \( y \) so there is no problem here. But
\[
\frac{\partial f}{\partial y}(t, y) = \frac{1}{2\sqrt{y}}
\]
does not exist when \( y = 0 \) (i.e., along the \( t \)-axis in the \( ty \)-plane) so it is not continuous there. The initial condition \( t_0 = 0, y_0 = 0 \) lies on the line \( y = 0 \), therefore \( \partial f / \partial y \) is not continuous on any open rectangle containing the point \((0, 0)\), and the hypotheses of Theorem 2.4.2 are not satisfied.

5. The function
\[
f(t, y) = y^3
\]
is continuous everywhere, and so is
\[
\frac{\partial f}{\partial y}(t, y) = 3y^2.
\]
Therefore the hypotheses of Theorem 2.4.2 are satisfied on any open rectangle in the \( ty \)-plane, and the initial value problem
\[
y' = y^3, \quad y(0) = y_0
\]
has a unique solution \( y = \phi(t) \) for any \( y_0 \). The solution is defined on some open interval that contains \( t = 0 \), but the theorem gives no information on how large this interval is.

To find the solution (and the open interval of definition) we note that the nonlinear differential equation is separable, write it as
\[
y^{-3} \, dy = dt,
\]
Integrating (and a little algebra) gives
\[
y^{-2} = -2t + c,
\]
where \( c \) is an arbitrary constant, and the initial condition at \( t = 0 \) gives
\[
c = y_0^{-2}
\]
provided that \( y_0 \neq 0 \).

Solving for \( y \) we get
\[
y = \pm \frac{1}{\sqrt{(y_0^{-2}) - 2t}}.
\]
If \( y_0 > 0 \) then we have to choose the + sign to satisfy the initial condition and
\[
y = \phi(t) = \frac{1}{\sqrt{(y_0^{-2}) - 2t}} \quad (y_0 > 0),
\]
which has interval of definition
\[-\infty < t < \frac{1}{2y_0}.\]
If \( y_0 < 0 \) then we must choose the \(-\) sign so that \( y(0) \) is negative,

\[
y = \phi(t) = -\frac{1}{\sqrt{(y_0^2 - 2t)}} \quad (y_0 < 0),
\]

which has the same interval of definition

\(-\infty < t < \frac{1}{2y_0^2}.\)

If \( y_0 = 0 \) then no value of \( c \) in the expression \( y = \pm 1/\sqrt{-2t + c} \) can be chosen to satisfy the initial condition \( y(0) = 0 \) (the expression \( y = \pm 1/\sqrt{-2t + c} \) is not a general solution), but one can easily verify ("guess and check") that

\[
y = \phi(t) = 0 \quad (y_0 = 0),
\]

with interval of definition

\(-\infty < t < \infty\)

solves the initial value problem.

6. Graph of \( f(y) \) versus \( y \): see Figure 1. (Notice that \( f'(y) = 3y^2 - 1 = 0 \) when \( y = -1/\sqrt{3} \) and \( y = 1/\sqrt{3} \). These correspond to the local maximum and local minimum in Figure 1.)

![Figure 1: Graph of \( f(y) = y^3 - y \) versus \( y \).](image)

The equilibrium solutions (critical points) are the solutions of

\[
0 = y^3 - y = (y^2 - 1)y = (y + 1)(y - 1)y
\]

which are

\[
y = -1, 0, 1.
\]
By observing the sign of \( y^3 - y \) in the various intervals of the \( y \)-axis (see below), we obtain the phase portrait in the phase line in Figure 2, and the classification of the equilibrium solutions (critical points):

\[
\begin{align*}
-1 & \text{ is unstable} \\
0 & \text{ is asymptotically stable} \\
1 & \text{ is unstable}
\end{align*}
\]

Figure 2: Phase line of \( y' = y^3 - y \).

To help sketch graphs of solutions in the \( ty \)-plane, we first note that \( y = -1, 0, 1 \) represent three solutions with graphs that are horizontal lines. For the other solutions, we consider whether the solution \( y(t) \) is increasing or decreasing, determined by the sign of \( f(y) \) (we could also predict the concavity of the graphs, determined by the sign of \( f(y)f'(y) \)).

a. If \(-\infty < y < -1\), then \( y(t) \) is decreasing, because \( f(y) < 0 \).

b. If \(-1 < y < 0\), then \( y(t) \) is increasing, because \( f(y) > 0 \).

c. If \( 0 < y < 1 \), then \( y(t) \) is decreasing, because \( f(y) < 0 \).

d. If \( 1 < y < \infty \), then \( y(t) \) is increasing, because \( f(y) > 0 \).

(see Figure 3, next page)
Figure 3: Several integral curves (graphs of solutions) of $y' = y^3 - y$ in the $ty$-plane.