Let us now return to the general nonlinear system, which we write in the scalar form

\[
\begin{align*}
X' &= F(x, y) \\
Y' &= G(x, y)
\end{align*}
\]  
\[--- (5)\]

Theorem 9.2.2

The system (5) is locally linear in the neighborhood of a critical point \((x_0, y_0)\) whenever the functions \(F\) and \(G\) have continuous derivatives up to order two.

Provided by this theorem, the corresponding linear system at point \(\text{critical} \ (x_0, y_0)\) is

\[
\frac{d}{dt} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} F_x(x_0, y_0) & F_y(x_0, y_0) \\ G_x(x_0, y_0) & G_y(x_0, y_0) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \]  
\[--- (6)\]

where \(u_1 = x - x_0\) and \(u_2 = y - y_0\).

Equation (6) provides a simple and general method for finding the linear system corresponding to a locally linear system near a given critical point.
The matrix

\[ J = J \begin{bmatrix} F & G \\ G_x & G_y \end{bmatrix}(x, y) = \begin{pmatrix} F_x(x, y) & F_y(x, y) \\ G_x(x, y) & G_y(x, y) \end{pmatrix} \]

is called the Jacobian matrix of the function \( F \) and \( G \) with respect to \( x \) and \( y \).

**Example:** (consider small damping case, namely \( \gamma^2 < 4\omega^2 \))

The motion of a pendulum is described by the system

\[
\begin{align*}
\frac{dx}{dt} &= y \\
\frac{dy}{dt} &= -\omega^2 \sin x - \gamma y
\end{align*}
\]

\[ F(x, y) = y \]
\[ G(x, y) = -\omega^2 \sin x - \gamma y \]

\[ F_x(x, y) = 0, F_y(x, y) = 1 \]
\[ G_x(x, y) = -\omega^2 \cos x, G_y(x, y) = -\gamma \]

**Critical points**

\[
\begin{cases}
y = 0 \\
-\omega^2 \sin x - \gamma y = 0
\end{cases}
\]

\[ \Rightarrow \]

\[ \begin{cases}
x = n \pi, n \in \mathbb{Z} \\
y = 0
\end{cases} \]

Corresponding linear system at critical point \((n \pi, 0)\)

\[
\frac{d}{dt} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} F_x(x, y) & F_y(x, y) \\ G_x(x, y) & G_y(x, y) \end{pmatrix} \begin{pmatrix} x = n \pi \\ y = 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}
\]

\[ u_1 = x - n \pi \]
\[ u_2 = y \]

9-3 - (4)
\[
\frac{d}{dt} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -w^2 \cos n\pi & -\gamma \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}
\]

\[u_1 = x - n\pi, \quad u_2 = y,\]

For \( n \) is even, \((n = 2k)\)
\[
\frac{d}{dt} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -w^2 & -\gamma \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}
\]

Eigenvalues
\[
\begin{vmatrix} -r & 1 \\ -w^2 & -\gamma - r \end{vmatrix} = 0 \Rightarrow r^2 + \gamma \cdot r + w^2 = 0
\]

\[\Rightarrow \lambda_{1,2} = -\frac{\gamma \pm \sqrt{\gamma^2 - 4w}}{2},\]

Recall that \( \gamma^2 < 4w \)

\[\Rightarrow \lambda_{1,2} = -\gamma \pm i w\]

hence the critical point \((2k\pi, 0)\) is a spiral and it's asymptotically stable.

For \( n \) is odd, \((n = 2k+1)\)
\[
\frac{d}{dt} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -w^2 & -\gamma \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}
\]

Eigenvalues
\[
\begin{vmatrix} -r & 1 \\ -w^2 & -\gamma - r \end{vmatrix} = 0 \Rightarrow r^2 + \gamma \cdot r - w^2 = 0
\]

\[\Rightarrow \lambda_{1,2} = -\frac{\gamma \pm \sqrt{\gamma^2 + 4w}}{2},\]

\(\gamma\) is positive, \(\lambda_2\) is negative

hence the critical point \((c2\pi, 0)\) is a saddle point
locally, the trajectories around \((2\pi, 0)\) looks like

Therefore, the trajectories of the pendulum with small damping looks like