Basic theory of system of first-order linear equation

**Matrix form**

\[
\begin{align*}
X_1' &= \rho_{11}(t) X_1 + \cdots + \rho_{1n}(t) X_n + g_1(t), \\
X_2' &= \rho_{21}(t) X_1 + \cdots + \rho_{2n}(t) X_n + g_2(t), \\
&\vdots \\
X_n' &= \rho_{n1}(t) X_1 + \cdots + \rho_{nn}(t) X_n + g_n(t)
\end{align*}
\]  

(1)

Let

\[
X(t) = \begin{pmatrix} 
X_1(t) \\
X_2(t) \\
\vdots \\
X_n(t)
\end{pmatrix}; \quad \rho(t) = \begin{pmatrix} 
\rho_{11}(t) & \cdots & \rho_{1n}(t) \\
\rho_{21}(t) & \cdots & \rho_{2n}(t) \\
\vdots & \ddots & \vdots \\
\rho_{n1}(t) & \cdots & \rho_{nn}(t)
\end{pmatrix}
\]

\[
\mathbf{g}(t) = \begin{pmatrix} 
g_1(t) \\
g_2(t) \\
\vdots \\
g_n(t)
\end{pmatrix}
\]

then eq system (1) can be rewritten as

\[
X' = \rho(t) X + \mathbf{g}(t)
\]  

(2)
Notation of Solutions

\[ X^{(1)}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}; \quad X^{(k)}(t) = \begin{pmatrix} x_1^{(k)}(t) \\ x_2^{(k)}(t) \\ \vdots \\ x_n^{(k)}(t) \end{pmatrix} \]

First Solution of \( (2) \)

\[ X^{(k)}(t) \]

\[ X^{(k)}(t) \]

k-th Solution of \( (2) \)

Consider the homogeneous system

\[ X' = p(t) X \]  \( \quad \text{(3)} \)

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Theorem 7.4.1: Principle of Superposition

If the vector functions \( X^{(1)} \) and \( X^{(2)} \) are solutions of System \( (3) \), then the linear combination \( c_1 X^{(1)} + c_2 X^{(2)} \) is also a solution of \( (3) \) for any constants \( c_1 \) and \( c_2 \).

Generally

If \( X^{(1)}, X^{(2)}, \ldots, X^{(n)} \) are solutions of System \( (3) \), the the linear combination \( c_1 X^{(1)} + c_2 X^{(2)} + \ldots + c_n X^{(n)} \) is also a solution of \( (3) \) for any constants \( c_1, \ldots, c_n \). 7.4-2
Wronskian determinant

For the solutions $x^{(1)}, x^{(2)}, \ldots, x^{(n)}$
the Wronskian determinant is

$$W[x^{(1)}, \ldots, x^{(n)}] = \det \begin{pmatrix} x^{(1)} & \ldots & x^{(n)} \end{pmatrix} = \det \begin{pmatrix} x^{(1)} \end{pmatrix}$$

The solutions $x^{(1)}, \ldots, x^{(n)}$ are then linearly independent at a point $x_0$
if and only if $W[x^{(1)}, \ldots, x^{(n)}](x_0) \neq 0$.

Connection with the second order equation.

For $y'' + p(x)y' + q(x)y = 0$

Let $x_1 = y$, $x_2 = y'$, then corresponding

System is

\[
\begin{align*}
x_1' &= x_2 \\
x_2' &= -p(x_1) x_1 - q(x) x_2
\end{align*}
\]

Wronskian for second order equation defined as

\[
W[x_1, x_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x_1^{(1)} & x_2^{(1)} \\ x_1^{(2)} & x_2^{(2)} \end{vmatrix} = W[x_1, x_2]
\]

Wronskian for the system and second order equation
are the same!

7.4 - 3
Consider the matrix $X(t)$ whose columns are vectors $x^{(1)}(t), \ldots, x^{(n)}(t)$

$$X(t) = \begin{pmatrix}
X_{11}(t) & X_{12}(t) & \cdots & X_{1n}(t) \\
X_{21}(t) & X_{22}(t) & \cdots & X_{2n}(t) \\
\vdots & \vdots & \ddots & \vdots \\
X_{n1}(t) & X_{n2}(t) & \cdots & X_{nn}(t)
\end{pmatrix}$$

Wronskian of the $n$-solutions $x^{(1)}, \ldots, x^{(n)}$ defined as

$$W[x^{(1)}, \ldots, x^{(n)}](t) = \det X(t)$$

The solutions $x^{(1)}, \ldots, x^{(n)}$ are then linearly independent at a point if and only if

$$W[x^{(1)}, \ldots, x^{(n)}](t) \neq 0$$

Thus: If the vector functions $x^{(1)}, \ldots, x^{(n)}$ are linearly independent solutions of $X' = p(t)X$ for each point in the interval $a \leq t \leq b$, then each solution $x = x(t)$ of $X' = p(t)X$ can be expressed as a linear combination of $x^{(1)}, \ldots, x^{(n)}$

$$x(t) = c_1 x^{(1)} + \cdots + c_n x^{(n)}$$

in exactly one way.
Thus consider system

\[ X' = p(t)X \]

where each element of \( p \) is a real valued continuous function. If \( x = u(t) + iv(t) \) is a complex-valued solution of equation 31, then its real part \( u(t) \) and its imaginary part \( v(t) \) are also solutions of the equation.