chapter 7

Systems of First-order Linear Equations

Rewrite a second order differential equation as a system of first order equations.

Example:

\[ u'' + \frac{1}{8} u' + u = 0 \]  \hspace{1cm} (1)

Let \( x_1 = u \), \( x_1(t) = u(t) \), \( x_2 = u' \) \( (x_2(t) = u'(t)) \)

Then:

\[ x_1' = u' = x_2 \quad x_2' = u'' \]

plugging \( u'' = x_2' \) and \( u' = x_1' \), \( u = x_1 \) into the equation \( (1) \), we get:

\[ x_2' + \frac{1}{8} x_2 + x_1 = 0 \]

\[ \Rightarrow x_2' = -\frac{1}{8} x_2 - x_1 \]

The first equation is

\[ x_1' = x_2 \]

Corresponding system:

\[ \begin{align*}
    x_1' &= x_2 \\
    x_2' &= -\frac{1}{8} x_2 - x_1
  \end{align*} \]
general system

\begin{align*}
    x_1' &= F_1(t, x_1, x_2, \ldots, x_n) \\
    x_2' &= F_2(t, x_1, x_2, \ldots, x_n) \\
    &\vdots \\
    x_n' &= F_n(t, x_1, x_2, \ldots, x_n)
\end{align*} \quad (2)

\text{initial condition}

\begin{align*}
    x_1(t_0) &= x_1^0, \quad x_2(t_0) = x_2^0, \quad \ldots, \quad x_n(t_0) = x_n^0. \quad (3)
\end{align*}

\text{initial value problem: (2), and (3),}

\underline{Existence and uniqueness Theorem: Nonlinear case}

Theorem 7.1.1

1) $N$ functions $F_1, \ldots, F_n$ and $n^2$ first partial derivatives
   \[ \frac{\partial F_1}{\partial x_1}, \ldots, \frac{\partial F_1}{\partial x_n}, \ldots, \frac{\partial F_n}{\partial x_1}, \ldots, \frac{\partial F_n}{\partial x_n} \]
   be continuous
   in a region $R$ of $t \times x_1 \times \ldots \times x_n$-space defined by
   $a \leq t \leq b$, $\alpha \leq x_1 \leq \beta_1$, $\ldots$, $\alpha \leq x_n \leq \beta_n$.

2) point $(t_0, x_1^0, x_2^0, \ldots, x_n^0) \in R$ (initial point)

Then there is an interval $1-t_0| < \delta$ in which there exists a unique solution
   $x_1 = \phi_1(t), \ldots, x_n = \phi_n(t)$
   satisfies (2), and (3).
If \( F_1, F_2, \ldots, F_n \) in (2) are linear functions of the independent variable \( x_1, \ldots, x_n \), then the system is said to be linear. Thus (2) could be written as

\[
\begin{align*}
\dot{x}_1 &= p_{11}(t) x_1 + \cdots + p_{1n}(t) x_n + g_{1}(t), \\
\dot{x}_2 &= p_{21}(t) x_1 + \cdots + p_{2n}(t) x_n + g_{2}(t), \quad \ldots \quad (4) \\
\dot{x}_n &= p_{n1}(t) x_1 + \cdots + p_{nn}(t) x_n + g_{n}(t).
\end{align*}
\]

Further if all \( g_j(t) \), \( j = 1, \ldots, n \) are zero in interval \( I \), then the system \((4)\) is said to be homogeneous; otherwise, it is nonhomogeneous.

Existence and uniqueness: Linear case

**Theorem 7.1.2**

1) \( n \times n \) function \( p_{ij} \), \( 1 \leq i, j \leq n \) are continuous on an open interval \( I: \alpha < t < \beta \).

2) \( t \in I \)

Then there exists a unique solution \( x_1 = \phi_1(t), \ldots, x_n = \phi_n(t) \) satisfies system \((4)\) and initial condition \( \beta \), here \( x_0, \ldots, x_0 \) could be any prescribed numbers. And the solution exists throughout the interval \( I \).