3.7 Mechanical Vibrations

The equation of motion of the mass is
\[ m\ddot{u}(t) + \gamma \dot{u}(t) + ku(t) = f(t). \]

\( m \): mass
\( \gamma \): is a positive constant of proportionality known as the damping constant.
\( k \): the spring constant
the constants \( m, \gamma \) and \( k \) are positive.
\( f(t) \): external force
\( \left\{ \begin{array}{c}
directed \text{ downward} \\
downward \\
\text{ upward} \\
\end{array} \right. \)

The complete formulation of the vibration problem requires two initial conditions
\( u(0) = u_0 \), the initial position of the mass.
\( u'(0) = v_0 \), the initial velocity
Example: (No damping), (No external force)
Suppose that a mass weighing 10 lb stretches a spring 2 in. If the mass is displaced an additional 2 in and is then set in motion with an initial upward velocity of 1 ft/s, determine the position of the mass at any latter time. Also determine the period, amplitude and phase of the motion.

Solution:

The spring constant is \( k = \frac{10 \text{ lb}}{2 \text{ in}} = 60 \text{ lb/ft} \) \[ (1 \text{ in} = \frac{1}{12} \text{ ft}) \]

the mass \( m = \frac{w}{g} = \frac{10 \text{ lb}}{32 \text{ ft/s}^2} = \frac{10}{32} \text{ lb. s}^2/\text{ft} \]

No damping, No external force.
Hence the equation of motion reduces to
\[
\frac{10}{32} u'' + 60 u = 0 \quad \Rightarrow \quad u'' + 192 u = 0 \quad - (1)
\]
The initial conditions
\[
u(0) = 2 \text{ in} = \frac{1}{6} \text{ ft} \\
u'(0) = 1 \text{ ft/s}
\]
The characteristic equation of (1) 
\[ r^2 + 192 = 0, \quad r = \pm 8\sqrt{3} \imath, \]
hence the general sol of (1) is 
\[ u(t) = A \cos(8\sqrt{3} t) + B \sin(8\sqrt{3} t). \]

To satisfy the initial conditions, 
\[ u(0) = \frac{1}{6} \implies A = \frac{1}{6} \]
\[ u'(0) = -1, \quad u'(t) = -8\sqrt{3} A \sin(8\sqrt{3} t) + 8\sqrt{3} B \cos(8\sqrt{3} t) \]
\[ \implies 8\sqrt{3} B = -1 \implies B = -\frac{1}{8\sqrt{3}} \]
Therefore, the solution for initial value problem (1), (2) is 
\[ u(t) = \frac{1}{6} \cos(8\sqrt{3} t) - \frac{1}{8\sqrt{3}} \sin(8\sqrt{3} t) \quad (4) \]

To determine the period, amplitude, let's consider the general function 
\[ A \cos(\omega t) + B \sin(\omega t), \]
we have two formulas about 
\[ A \cos(\omega t) + B \sin(\omega t), \]
Formula 1: 
\[ A \cos(\omega t) + B \sin(\omega t) = \sqrt{A^2 + B^2} \cos(\omega t - \alpha), \]
where \( \alpha = \arctan \frac{B}{A}, \) (\( \alpha \) minus sign)
Formula 2: 
\[ A \cos(\omega t) + B \sin(\omega t) = \sqrt{A^2 + B^2} \sin(\omega t + \delta) \]
where \( \delta = \arctan \frac{A}{B}, \)

3.7-3
By formula 1 or formula 2, we can see that the period and amplitude of the function $A \cos(\omega t) + B \sin(\omega t)$ are

$$\sqrt{A^2 + B^2} \approx \frac{2\pi}{\omega}, \sqrt{A^2 + B^2} \text{ respectively.}$$

Back to this case,

$A = \frac{1}{6}, \ B = -\frac{1}{8\sqrt{3}}, \ \omega = 8\sqrt{3}$

Period: $\frac{2\pi}{\omega} = \frac{2\pi}{8\sqrt{3}} = \frac{\pi}{4\sqrt{3}}$

Amplitude: $\sqrt{A^2 + B^2} = \sqrt{\frac{1}{36} + \left(-\frac{1}{8\sqrt{3}}\right)^2} = \sqrt{\frac{19}{576}} \approx 0.1820$

$\theta = \arctan \frac{B}{A} = \arctan \frac{-\frac{1}{8\sqrt{3}}}{\frac{1}{6}}$

$\approx \arctan (-\frac{\sqrt{3}}{4}) = -\arctan \frac{\sqrt{3}}{4}$

$\approx \pi - 0.40864$

The graph of solution in (4) = $\sqrt{\frac{19}{576}} \cos(8\sqrt{3}t + 0.409)$

$\theta = \frac{-0.409}{8\sqrt{3}} \Rightarrow \text{Here the textbook is wrong!}$
Example 2: (With damping, No external force)

The motion of a certain spring-mass system is governed by the differential equation
\[ u'' + \frac{1}{8} u' + u = 0 \]  
(1)

where \( u \) is measured in feet and \( t \) in seconds.

\[ u(0) = 2, \ u'(0) = 0 \]  
(2)

determine the position of the mass at any time.

Solution:

Characteristic equation of (1) is
\[ r^2 + \frac{1}{8} r + 1 = 0 \]

\[ r = -\frac{1}{16} \pm \frac{\sqrt{255}}{16} i \]

hence the general solution of (1) is

\[ u(t) = e^{-\frac{t}{16}} \left( A \cos \frac{\sqrt{255}}{16} t + B \sin \frac{\sqrt{255}}{16} t \right) \]

\[ u'(t) = -\frac{1}{16} A e^{-\frac{t}{16}} \cos \frac{\sqrt{255}}{16} t - \frac{\sqrt{255}}{16} A e^{-\frac{t}{16}} \sin \frac{\sqrt{255}}{16} t \]

\[ -\frac{1}{16} B e^{-\frac{t}{16}} \sin \frac{\sqrt{255}}{16} t + \frac{\sqrt{255}}{16} B e^{-\frac{t}{16}} \cos \frac{\sqrt{255}}{16} t \]

\[ u(0) = 2 \implies A = 2 \]

\[ u'(0) = 0 \implies -\frac{1}{16} A + \frac{\sqrt{255}}{16} B = 0 \implies B = \frac{2}{\sqrt{255}} \]
Hence, the solution of the initial value problem (IVP) \( u(1), u_2 \) is

\[
\begin{align*}
\dot{u}(t) &= e^{-\frac{t}{16}} \left( 2 \cos \frac{255}{16} t + \frac{2}{\sqrt{255}} \sin \frac{255}{16} t \right) \\
&= e^{-\frac{t}{16}} \sqrt{2 + \left( \frac{3}{255} \right)^2} \cos \left( \frac{255}{16} t - \theta \right) \\
&= e^{-\frac{t}{16}} \frac{3}{\sqrt{255}} \cos \left( \frac{255}{16} t - \theta \right) \quad (3)
\end{align*}
\]

where \( \theta = \arctan \frac{B}{A} = \arctan \frac{1}{255} \approx 0.06254 \)

Graph of the solution in (3)

The amplitude decreases and converges to zero as \( t \to \) and \( t \to +\infty \), this is what the damping means.

If no damping, the amplitude conserves as \( t \) increases.

2.8-6
General discussion for the case damped and free vibration (no external force).

The equation damped and free vibration has the following form:

\[ m u'' + \gamma u' + ku = 0 \]

\[ \downarrow \quad \text{damped free vibration} \]

The corresponding characteristic equation is

\[ mr^2 + \gamma r + k = 0 \]

The roots are

\[ r_{1,2} = -\frac{\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m} \]

Depending on the sign of the discriminant \( \gamma^2 - 4mk \), the solution \( u \) has one of the form

\[
\begin{align*}
\gamma^2 - 4mk > 0, & \quad u = Ae^{rt} + Be^{rt} \quad (5) \\
\gamma^2 - 4mk = 0, & \quad u = (A + Br)e^{rt} \quad (6) \\
\gamma^2 - 4mk < 0, & \quad u = e^{-\frac{rt}{2m}}(A\cos rt + B\sin rt) \quad (7)
\end{align*}
\]

\[ u = \frac{1}{2m}(4km - \gamma^2)^{1/2} > 0 \]

3.7 (7)
For the graph of (5), it's a combination of two exponential functions.

For the graph of (7), (Recall that \(v, w, k\) all are positive), we have discussed in Example 2 (damped).

For the function (6), for example, the function

\[ u(t) = (\frac{1}{2} + 2t) e^{-\frac{t}{2}} \]

like the exponential function

like a linear function

for \(B\) is negative, for example

\[ u(t) = (\frac{1}{2} - \frac{3}{2}t) e^{-\frac{t}{2}} \]

like a linear function

like an exponential function