Example: Consider the initial value problem
\[ t^2 y'' + q + y' + 16y = 0, \quad y(1) = -1, \quad y'(1) = 1. \]

(a) Determine the largest exist interval of the initial value problem? (=>) Determine on what open interval the solution \( y = \phi(t) \) of the initial value problem is defined?

(b) Verify \( y(t) = t^{-4} \) is a solution of the ODE. Find another solution \( y(t) \).

(c) Verify that solutions \( y_1 \) and \( y_2 \) form a fundamental set of solutions. On what open intervals?

(d) Solve the initial value problem.
(a): (The longest exist interval) by the uniqueness existence theorem of second order ODE (Th 3.2.1).

Reform the equation to the standard form (which means the coefficient of the second order derivative is constant 1)

\[ y'' + \frac{9}{t} y' + \frac{16}{t^2} y = 0 \]

where \( p(t) = \frac{9}{t} \), \( q(t) = \frac{16}{t^2} \).

find the continuous interval of \( p(t) \) and \( q(t) \),

we denote as \( C_{p(t)} \), \( C_{q(t)} \) respectively.

\( C_{p(t)} : t > 0 \) or \( t < 0 \)

\( C_{q(t)} : t > 0 \) or \( t < 0 \)

The intersection of continuous interval of \( p(t) \) and \( q(t) \), namely,

\[ C_{p(t)} \cap C_{q(t)} = \{ t / t > 0 \} \text{ or } \{ t / t < 0 \} \]

Since the initial condition given at the point 1, then the interval desired is \( \{ t / t > 0 \} \).

Remark: The longest interval is open and "continuous" (which means no interrupt point).

3.4 ②
(b): \( y(t) = t^{-4} \) is solution of the ODE since
\[ y_1' = -4t^{-5}, \ y_1'' = 20t^{-6}, \ \text{and} \ y_1'' + 9ty_1' + 16y_1 = 0. \]

To find another solution of the ODE, we use the method "Reduction of Order".

Step 1: Let \( y(t) = y_1(t) w(t) \), plugging it into the equation where \( y_1(t) \) is the solution we have known, and \( w(t) \) is to be determined.

Then \( y' = y_1'w + y_1w' \); \( y'' = y_1''w + 2y_1'w' + y_1w'' \)

\[ 0 = ty'' + 9ty' + 16y_1w \]
\[ = t^2(y_1''w + 2y_1'w' + y_1w'') + 9t(y_1'w + y_1w') + 16y_1w \]
\[ = t^2(20t^{-6}w - 8t^{-5}w' + t^{-4}w'') + 9t(-4t^{-5}w + t^{-4}w') + 16t^{-4}w \]
\[ = t^{-2}w'' + t^{-3}w', \]

Step 2: Solve the equation for \( w(t) \).

\[ t^{-2}w'' + t^{-3}w' = 0 \quad (\Rightarrow \ w'' + \frac{1}{t}w' = 0) \]

Let \( v = w' \), then (1) becomes

\[ v' + \frac{1}{t}v = 0 \] (This is the origin of the Reduction of order.)

Note that eq (1) is a Second order ODE, and eq (2) is a First order ODE.
To solve (2), the integrating factor of equation (2) is \( \mu (t) = e^{\int \frac{1}{t} dt} = e^{\ln t + c} = ct \). Take \( \mu (t) = e^t \) (by letting in the above expression, this is the simplest case of \( \mu (t) \)). Multiply eq (2) by \( t \) on both sides, then we get

\[
tv' + v = 0 \quad \Rightarrow \quad (tv)' = 0
\]

\[
= \quad tv = c_1, \quad v = \frac{c_1}{t}.
\]

Recall that \( v = w' \), hence

\[
w' = \frac{c_1}{t} \quad \Rightarrow \quad w = c_1 \ln t + c_2, \quad (since \ t > 0)
\]

Therefore, \( y_1 (t) = y_1 \cdot w = t^{-4} (c_1 \ln t + c_2) \)

\[
= c_1 \cdot t^{-4} \ln t + c_2 \cdot t^{-4}
\]

In particular, another solution \( y_2 (t) \) could be \( t^{-4} \ln t \).

To verify \( y_1 \) and \( y_2 \) form a fundamental set of solution of the ODE, we only need to check the Wronskian of \( y_1, y_2 \) is non-zero.

\[
W(y_1, y_2) = \begin{vmatrix}
y_1 & y_2 \\
y_1' & y_2'
\end{vmatrix} = \begin{vmatrix} t^{-4} & t^{-4} \ln t \\
-4t^{-5} & -4t^{-5} \ln t + t^{-5}
\end{vmatrix} = t^{-9} \neq 0
\]

(\text{since } t > 0)
(d): By (c) The general solution of the ODE can be written as
\[ y(t) = C_1 y_1 + C_2 y_2 \]
\[ = C_1 t^{-4} + C_2 t^{-4} \ln t \]
thus \[ y' = -4C_1 t^{-5} - 4C_2 t^{-5} \ln t + C_2 t^{-5} \]
By the initial condition, we can get two linear equations about \( C_1, C_2 \)
\[ y(1) = -1 \implies \{ \begin{array}{c} C_1 = -1 \\ C_2 = -3 \end{array} \] \]
\[ y'(1) = 1 \implies -4C_1 + C_2 = 1 \]
Therefore the solution of the initial value problem is
\[ y(t) = -t^{-4} - 3 t^{-4} \ln t. \]

Summary: Given one solution \( y_1(t) \), to find another solution \( y_2(t) \), we let the solution have the form \( y_2(t) = W(t) \), where \( W(t) \) is to be determined, then we get a new equation \( \frac{dW}{dt} \) with respect to \( W(t) \), the equation for \( W(t) \) could be one order less than the original equation by letting \( u = W \).
We continue the discussion of the Second ODE with constant coefficients.

\[ ay'' + by' + cy = 0 \quad (2) \]

where \( a, b, c \) are given constants (real) and \( a \neq 0 \).

The corresponding characteristic equation is

\[ \lambda^2 + br + c = 0 \quad (3) \]

\[ \Delta = b^2 - 4ac \]

Case 1: \( \Delta > 0 \), then (3) admits two distinct real roots.

We denote these as \( \lambda_1, \lambda_2 \), then the general solution of (2) is

\[ y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \quad \text{(Section 3.1)} \]

Case 2: \( \Delta < 0 \), then (3) admits two conjugate complex roots, these complex roots have the form

\[ \lambda = \alpha + \beta i \]

then the general solution of (2) is

\[ y(t) = c_1 e^{\alpha t} \cos(\beta t) + c_2 e^{\alpha t} \sin(\beta t) \]

Case 3: \( \Delta = 0 \), then (3) admits repeated roots (real) namely, \( \lambda_1 = \lambda_2 \), \( (\lambda_1, \lambda_2 \) are real), then the general solution of (2) is

\[ y(t) = c_1 e^{\lambda_1 t} + c_2 t e^{\lambda_1 t} \]

\[ 3.4 \, \text{6} \]
Remark: The Theorem above we can apply directly without proof when we solve a initial value problem.

2: For the case 1, case 2, we have proved in Section 3.1 and 3.3 respectively, the proof of case 3 will be given better below.

The method is the "Reduction of order".

Example: \( y'' + 2y' + y = 0 \) — (4)

The characteristic equation is

\[ r^2 + 2r + 1 = 0 \]

So \( r_1 = r_2 = -1 \), then we can find one solution of eq (4), \( y_1 = e^{rt} = e^t \) (\( e^{rt} \) gives the solution as \( y_1(t) \)).

Question: How can we find another solution of equation (4)?
Let $y(t) = y_1(t) \cdot \text{W}(t)$, find the new equation for $\text{W}(t)$. ($y_1(t) = e^{-t}$)

$y' = y_1' \cdot \text{W} + y_1 \cdot \text{W}' = -e^{-t} \cdot \text{W} + e^{-t} \cdot \text{W}'$

$y'' = y_1'' \cdot \text{W} + 2y_1' \cdot \text{W}' + y_1 \cdot \text{W}'' = e^{-t} \cdot \text{W} - 2e^{-t} \cdot \text{W}' + e^{-t} \cdot \text{W}''$

Plugging into the equation (4),

$0 = y'' + 2y' + y$

$= (e^{-t} \cdot \text{W} - 2e^{-t} \cdot \text{W}' + e^{-t} \cdot \text{W}'')$

$+ 2(-e^{-t} \cdot \text{W} + e^{-t} \cdot \text{W}') + e^{-t} \cdot \text{W}$

$= e^{-t} \cdot \text{W}''$  \quad \Rightarrow  \quad e^{-t} \cdot \text{W}'' = 0$

$\Rightarrow \quad \text{W}'' = 0 \quad \Rightarrow \quad \text{W}' = c_1 \quad \Rightarrow \quad \text{W} = c_1 t + c_2$

(Since this can be viewed as $[\text{W}']' = 0$).

Recall $y(t) = y_1(t) \cdot \text{W}(t) = e^{-t} (c_1 t + c_2)$

$= c_1 t e^{-t} + c_2 e^{-t}$

Therefore, another solution can be $y_2(t) = t e^{-t}$.
To make sure the linear combination of $y_1, y_2$ is the general solution of eq (4), we need to verify that the Wronskian of $y_1, y_2$ is non-zero.

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} e^{-t} & t e^{-t} \\ -e^{-t} & e^{-t} - t e^{-t} \end{vmatrix} = e^{-2t} \neq 0$$

$W(y_1, y_2) \neq 0$, hence $y_1, y_2$ form a fundamental set of solution of equation (4) (namely, $y_1, y_2$ are the basis of solution space of eq (4)). Therefore, the general solution of can be represent as

$$y(t) = C_1 y_1(t) + C_2 y_2(t)$$

$$= C_1 e^{-t} + C_2 e^{-t}$$

The proof of case 3 is similar as the example above, we omit the proof (equivalently, the calculation; further we only need to apply this conclusion without the proof).