3.3 Complex Roots of
classical Equation: the case
\( A < 0 \).

We continue our discussion of
\[ ay'' + by' + c = 0 \quad \text{(1)} \]

The characteristic equation of above equation is
\[ ar^2 + br + c = 0 \quad \text{(2)} \]

The discriminant \( \Delta : = b^2 - 4ac \)

\( \Delta > 0 \): Then (2) admits two distinct Real Roots
we have discussed in Section 3.1.

\( \Delta = 0 \): We will be discussed in Section 3.4

\( \Delta < 0 \): Then (2) admits two conjugate complex
numbers, we denote by
\[ \lambda + i\mu, \quad \lambda - i\mu \]

Recall the above Section about the
Complex valued solutions.
The complex valued function $e^{(\lambda + i\mu)t}$ is a solution of the second linear homogeneous differential equation with constant coefficients (1).

$$e^{(\lambda + i\mu)t} = e^{\lambda t} \cos \mu t + i \cdot e^{\lambda t} \sin \mu t.$$  

Hence the real and complex part

$e^{\lambda t} \cos \mu t$, $e^{\lambda t} \sin \mu t$ are the solutions of (1).

The similar analysis to $e^{(\lambda - i\mu)t}$ gives us the same result above.

Therefore, for the homogeneous ODE with constant's coefficients

$$ay'' + by' + cy = 0 \quad a \neq 0, b, c \neq 0$$  

if $A > 0$, then there are exist two complex roots $\mu_1 = \lambda + i\mu$, then the solutions of the equation (1) are given by

$$y = c_1 e^{\lambda t} \cos \mu t + c_2 e^{\lambda t} \sin \mu t$$  

3.3 (1)
Remark: \( y_1 = e^{lt} \cos mt, \ y_2 = e^{lt} \sin mt \)

\[
W(y_1, y_2) = \begin{vmatrix}
    e^{lt} \cos mt & e^{lt} \sin mt \\
    le^{lt} \cos mt & le^{lt} \sin mt \\
    -le^{lt} \sin mt & te^{lt} \cos mt \\
\end{vmatrix} = \begin{align*}
    e^{lt} & \\
    = & \neq 0
\end{align*}
\]

Hence \( y_1, y_2 \) form a basis of fundamental set of solutions of \( 69(\pi) \), therefore

\[
y = c_1 y_1 + c_2 y_2 = c_1 e^{lt} \cos mt + c_2 e^{lt} \sin mt \]

is a general solution of \( 69(\pi) \).

Example: Find the solution of initial value problem

\[
16y'' - 8y' + 145y = 0, \quad y(0) = -2, \quad y'(0) = 1
\]

Step 1: Characteristic equation

\[
16r^2 - 8r + 145 = 0 \Rightarrow r_1, r_2 = \frac{1}{4} \pm \frac{iv}{3}
\]

\[
r = \frac{1}{4}, \quad m = 3
\]

thus the general solution of the ODE is given by

\[
y(t) = c_1 e^{\frac{t}{4}} \cos 3t + c_2 e^{\frac{t}{4}} \sin 3t
\]
Step 2: Determine the constants $C_1, C_2$

\[y(0) = -2, \quad C_1 = -2\]
\[y'(0) = 1, \quad \frac{1}{4} C_1 + 3C_2 = 1\]

hence $C_2 = \frac{1}{2}$,

thus the solution for the initial value problem is given by

\[y(t) = -2e^{\frac{1}{2}t}\cos 3t + \frac{1}{2}e^{\frac{1}{2}t}\sin 3t\]

Remark 3: The main purpose of this section is to know how to solve a initial value problem like Example 3.3.