2.5 Autonomous Equation
and Population Dynamics

An equation is called autonomous if has the form

$$\frac{dy}{dt} = f(y) \quad (1)$$

Equation (1) is separable, so the discussion in Section 2.2 is applicable to it, namely,

$$\frac{dy}{f(y)} = dt = \int \frac{dy}{f(y)} = t + c$$

The main purpose of this section is to show how geometrical methods can be used to obtain important qualitative information directly from the differential equation without solving the equation.

1. Exponential Growth

$$\frac{dy}{dt} = ry \quad (2)$$

where the constant of proportionality r is called the rate of growth/decline, depending on whether it is positive or negative.

Here we assume that $r > 0$.

Solving the equation (2) with the initial condition

$$y(0) = y_0 \quad (3)$$
2 Logistic Growth

To take account of the fact that growth rate actually depends on the population, we replace the constant \( r \) in eq. (2) by a function \( h(y) \) and thereby obtain the modified equation (eq. 5) \( y' = r - ay \)

\[
\frac{dy}{dt} = (r - ay) y \quad (6)
\]

Equation (6) is known as Verhulst equation or the logistic equation. It is convenient to write the logistic equation in the equivalent form

\[
\frac{dy}{dt} = r \left(1 - \frac{y}{\kappa}\right) y, \quad \kappa = \frac{r}{a} \quad (7)
\]

The constant \( r \) is called the intrinsic growth rate.

Equilibrium solutions of (7)

\[
r \left(1 - \frac{y}{\kappa}\right) y = 0 \quad \Rightarrow \quad y = 0, \ y = \kappa
\]

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Let $f(y) = r(1 - \frac{y}{K}) y$, the graph of $f(y)$.

**Hence**

If $0 < y < K$, $\frac{dy}{dt} > 0 \Rightarrow y$ increase

If $y > K$, $\frac{dy}{dt} < 0 \Rightarrow y$ decrease

Sketch the graphs of solutions of Eq. (7)

$K$, equilibrium solution

$\frac{K}{2}$, equilibrium solution

$t$

It is the upper bound that is approached, hence refer to $K$ as the saturation level or the environmental carrying capacity.
To solve eq (7),

\[ \frac{dy}{(1-y/k) y} = rd\,dt \]

(3) \( \left( \frac{1}{y} + \frac{1/k}{1-y/k} \right)dy = rd\,dt \)

Integrating on both sides, we obtain

\[ \ln|y| - \ln|1-\frac{y}{k}| = rt + c \]

where \( c \) is an arbitrary constant of integration to be determined from the initial condition \( y(0) = y_0 \).

\[ \frac{y}{1-y/k} = C_2 e^{rt} \], where \( C_2 = e^c \)

For \( y(0) = y_0 \), then \( c = \frac{y_0}{1-y_0/k} \), therefore,

\[ y = \frac{y_0 k}{y_0 + (k-y_0)e^{-rt}} \quad (11) \]

We have the solution for the initial value problem.

All the qualitative conclusions that we have reached earlier by geometrical reasoning can be confirmed by examining the solution (11).

For \( r > 0 \) and \( y_0 > 0 \)

\[ \lim_{t \to +\infty} y(t) = k \]
Thus, the constant solution \( \phi_2(t) = k \) is an asymptotically stable solution of Eq (7).

For the constant solution \( \phi_1(t) = 0 \), even solutions that start very near zero grow as \( t \) increases and, as we have seen, approach \( k \) as \( t \to \infty \). We say that \( \phi_1(t) = 0 \) is an unstable equilibrium solution.

A critical Threshold.

We now turn to a consideration of the equation

\[
\frac{dy}{dt} = -r(1 - \frac{y}{T})y
\]

(14)

where \( r \) and \( T \) are given positive constants.

Quantitative analysis

\[
0 < y < T, \quad \frac{dy}{dt} < 0, \quad y \text{ decreases}
\]

\[
y > T, \quad \frac{dy}{dt} > 0, \quad y \text{ increases}
\]
From this figure it appears that as time increases, y either approaches zero or grows with bound, depending on whether the initial value $y_0$ is less than or greater than $T$.

Thus $T$ is a threshold level.

In mathematical way,

If $y_0 < T$, then $\lim_{t \to \infty} y(t) = 0$.

If $y_0 > T$, then $\lim_{t \to \infty} y(t) = +\infty$, or $y(t) \to +\infty$ as $t \to \infty$.

Then $T$ is a threshold level.

The above conclusion can be confirmed by solving the equation (14) with the initial condition $y(0) = y_0$, namely,

$$y = \frac{y_0 T}{y_0 + (T-y_0)e^{-t}}$$
Logistic Growth with a Threshold.

We consider

\[ \frac{dy}{dt} = r(1 - \frac{y}{T})(1 - \frac{y}{K})y \]

where \( r > 0 \) and \( 0 < T < K \).

Equilibrium solution or critical points:

Let \( -r(1 - \frac{y}{T})(1 - \frac{y}{K})y = 0 \) ⇒ \( \begin{cases} y = 0 \\ y = T \\ y = K \end{cases} \)

1) \( 0 < y < T, \frac{dy}{dt} < 0, \) \( y \) decrease

2) \( T < y < K, \frac{dy}{dt} > 0, \) \( y \) increase

3) \( y > K, \frac{dy}{dt} < 0, \) \( y \) decrease

Asymptotically stable

Sketch graph of \( y(t) \)

Threshold level

\( \phi_1(t) = T \)

\( \phi_2(t) = K \)

\( \phi_3(t) = K \)

\( \phi_{\text{stable}} \)
A model of this general sort apparently describes the population of the passenger pigeon.