2.3 Modeling with First order equation

Differential equations are of interest to nonmathematicians primarily because of the possibility of using them to investigate a wide variety of problems in the physical, biological, and social sciences.

Three key steps.

Step 1: Construction of the Model.

Step 2: Analysis of the Model.

Step 3: Comparison with Experiment or Observation.

Example:

At time \( t = 0 \), a tank contains \( Q_0 \) lb of salt dissolved in 100 gal of water. Assume that water containing \( \frac{1}{10} \) lb of salt/gal is entering the tank at a rate of 1 gal/min and that the well-stirred mixture is draining from the tank at same rate. Set up the initial value problem that describes this flow process. Find out the amount of salt (q(t)) in the tank at any time, and also find the limiting amount of that is present after a very long time.
If \( r = 3 \) and \( Q_0 = 2Q \), find the time \( T \) after which the salt level is with 2% of \( Q \).

The rate of change of salt in the tank, \( \frac{dQ}{dt} \), is equal to the rate at which salt is flowing in minus the rate at which it is flowing out.

In symbols,

\[
\frac{dQ}{dt} = \text{rate in} - \text{rate out}
\]

\[ r \cdot \frac{Q}{4} = \frac{r}{4} \rightarrow r \cdot \frac{Q}{100}
\]

Hence

\[
\frac{dQ}{dt} = \frac{r}{4} - \frac{rQ}{100}, \quad r > 0 \quad (2)
\]

The initial condition is

\[ Q(0) = Q_0 \quad (3) \]

Rewrite the equation to the standard form of a linear equation, we have

\[
\frac{dQ}{dt} + \frac{rQ}{100} = \frac{r}{4}
\]

where \( r \) is a parameter. You can view it as a constant. The integrating factor is \( e^{\frac{rt}{100}} \) and the general solution is

\[ Q(t) = 25 + c e^{-\frac{rt}{100}} \]
where \( c \) is an arbitrary constant.

By the initial condition, we have

\[ c = Q_0 - 25. \]

Hence, the solution of the initial value problem (2.3) is

\[ Q(t) = 25 + (Q_0 - 25) e^{-\frac{t}{100}} \quad (6) \]

From equation (6), we have

\[ \lim_{t \to +\infty} Q(t) = 25. \]

Namely, the limiting value \( Q_L \) is 25, confirm our physical intuition.

For \( r = 3 \) and \( Q_0 = 2 \), \( Q_L = 50 \), then eq.(6) becomes

\[ Q(t) = 25 + 25 e^{-0.03 t} \quad (8) \]

Since 2% of \( Q_L = 25 \) is 0.5. Substituting \( t = T \) and

\[ Q(t) = 25 + 0.5 \]

in equation (8),

\[ 25 + 0.5 = 25 + 25 e^{-0.03 T} \Rightarrow T = \frac{\ln 0.5}{-0.03} \approx 130.4 \text{ (min)} \]