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The existence, uniqueness and nonexistence of the ground state to the \(N\)-coupled Schrödinger systems in \(\mathbb{R}^n (n \leq 4)\)*

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Abstract

Consider the following Schrödinger system:

\[
\begin{cases}
-\Delta u_j + \lambda_j u_j = \beta_{jj} u_j^3 + \sum_{i \neq j} \beta_{ij} u_i^2 u_j, & x \in \Omega, \\
u_j \geq 0, & x \in \Omega; \\
u_j = 0 & \text{on } \partial \Omega; \\
j = 1, ..., N,
\end{cases}
\]

where either \(\Omega = \mathbb{R}^n (n = 2, 3, 4)\) or \(\Omega \subset \mathbb{R}^4\) is a smooth bounded domain. Note that the cubic nonlinearities and the coupling terms are of critical growth whenever dimension \(n = 4\). We give a characterization of the least energy solutions when \(\Omega = \mathbb{R}^n (n = 2, 3)\) or \(\Omega \subset \mathbb{R}^4\) is a smooth bounded domain. Furthermore, when \(\Omega = \mathbb{R}^4\) and \(\lambda_j = 0, j = 1, 2, ..., N\), we obtain a nonexistence theorem about the least energy solutions provided attraction and repulsion coexist, i.e. some of \(\beta_{ij}, i \neq j\) are positive but some others are negative.

Keywords: existence, nonexistence, uniqueness, ground state solution, Schrödinger system

Mathematics Subject Classification numbers: 35B38; 35J10; 35J15; 35J20; 35J60; 49J40

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1. Introduction

Consider the steady-state $N$-coupled nonlinear Schrödinger system:

\[
\begin{aligned}
-\Delta u_j + \lambda_j u_j &= \beta_j u_j^3 + \sum_{k \neq j} \beta_k u_k^2 u_j, & \quad x \in \Omega; \\
u_j &= 0, x \in \Omega; & \quad u_j = 0 \quad \text{on } \partial \Omega; & \quad j = 1, \ldots, N,
\end{aligned}
\]  

(1.1)

where either $\Omega \subset \mathbb{R}^d$ is a smooth bounded domain or $\Omega = \mathbb{R}^n (n = 2, 3, 4)$. The motivation for studying (1.1) comes from the search for solitary wave solutions of the time-dependent $N$-coupled nonlinear Schrödinger equations given by

\[
\begin{aligned}
-\frac{1}{m} \Phi_j &= \Delta \Phi_j + \beta_j |\Phi_j|^2 \Phi_j + \sum_{i \neq j} \beta_i |\Phi_i|^2 \Phi_j, & \quad x \in \Omega, & \quad t > 0; \\
\Phi_j &= \Phi_j(x, t) \in \mathbb{C}, & \quad x \in \Omega, & \quad t > 0; \\
\Phi_j(x, t) &= 0, & \quad x \in \partial \Omega, & \quad t > 0; & \quad j = 1, \ldots, N.
\end{aligned}
\]  

(1.2)

When $n = 2, 3$, the system (1.2) appears in many physical and optical problems. Physically, the solution $\Phi_j$ denotes the $j$th component of the beam in Kerr-like photorefractive media. The positive constant $\beta_j$ stands for self-focusing in the $j$th component of the beam. The coupling constant $\beta_{ij} (i \neq j)$ describes the interaction between the $i$th and the $j$th components of the beam. When $\beta_j > 0$, the interaction is attractive; it is repulsive when $\beta_j < 0$. When $n = 1$, there are many analytic results on the solitary wave solutions to the $N$-coupled nonlinear Schrödinger system (1.2)—see, for example, [10, 19, 20, 22, 30]. From the physical experiment, when $n = 2$, the 2-component photorefractive screening solutions and self-trapped beam were observed (see [31]). It is natural to believe that for the higher dimensional case, the more component solitons and self-trapped beams exist, see Lin and Wei [24] for some results in Sirakov [37]. Among them, the ground state solution and its uniqueness are the important aspects to physicists.

Among other things, the uniqueness of the ground state solution to the system (1.1) is also an important and challenging topic to mathematicians. When $\dim n \leq 3$ and $N = 2$, Sirakov conjectured in [37] that, up to a translation, $(k \omega_3, \omega_3)$ is the unique positive solution to (1.1), where $\omega_3 \in \mathbb{R}$ is the unique positive radially symmetric solution of the problem $-\Delta u + \lambda u = u^3$ in $\mathbb{R}^n$ provided that $\lambda := \lambda_1 = \lambda_2$ and $k, l$ are the designated constants, viz. $k = \sqrt{\frac{\beta - \beta_0}{\beta - \beta_{11} \beta_{22}}}$, $l = \sqrt{\frac{\beta - \beta_{12}}{\beta - \beta_{11} \beta_{22}}}$, where $\beta := \beta_{12} = \beta_{21}$. Wei and Yao [43] proved this conjecture in case $\beta := \beta_{12} = \beta_{21} > \max \{\beta_{11}, \beta_{22}\}$ and in case $\beta > 0$ small enough. See also Ikoma [21], where the uniqueness of the positive solution is proved when the coupling constant is small enough. In particular, when $\dim n = 1$ and $N = 2$, the classification of the positive solutions is completely finished in [43]. The main tools of [21] and [43] are the ordinary differential equation (ODE) methods. In [9], Chen and Zou give a partial answer to Sirakov’s conjecture when $\beta$ is near the min $\{\beta_{11}, \beta_{22}\}$ from the left-hand side. Moreover, they obtain the asymptotic behavior and uniqueness of the least energy solutions in [9]. The uniqueness of the ground state was also studied by Ma and Zhao in [31] by an ODE method. In particular, the authors of [31] pointed out that their arguments cannot be applied to the $N$-coupled Schrödinger system with $N \geq 3$. Therefore, they leave the uniqueness of the ground state for the case of $N \geq 3$ as an open problem (see remark 8 of [31]). Chen and Zou [8] study the existence of ground state for the 2-coupled Schrödinger system in $\mathbb{R}^d$; they also obtain the asymptotic behavior of positive least energy solutions—see [16] for the asymptotic behavior of positive least energy solutions to the scalar equation. The first work establishing the existence of infinitely many positive solutions when $\lambda_1 \neq \lambda_2$ appeared in the manuscript by Guo...
and Wei [18]. In their celebrated paper [24], Lin and Wei consider the existence and nonexistence of the ground state for the $N$-coupled nonlinear Schrödinger equations in $\mathbb{R}^n (n \leq 3)$. For the 3-coupled Schrödinger equations in $\mathbb{R}^n (n \leq 3)$, Byeon et al [6] consider the asymptotic behavior of the positive least energy vector solution; multiple positive solutions were obtained by Sato and Wang in [36]; for a complete classification of ground-states for a 3-coupled nonlinear system see Liu and Wang in [29]. We also note that Liu et al in [28] consider the existence of nodal solutions for the $N$-coupled nonlinear Schrödinger systems in $\mathbb{R}^n (n = 2, 3)$.

In the case of $n \leq 3$, all the nonlinearities in (1.1) are of subcritical growth due to the Sobolev embedding. The existence of solutions has received a great deal of attention. We refer the reader to [1, 4, 5, 24, 32, 37] for the existence of the least energy solution, and to [15, 25, 26, 35] for the semiclassical states or singularly perturbed settings, and to [3, 14, 41, 42] for the existence of multiple solutions. In particular, the authors deal with the system (1.1) in [12] when $1 \leq N \leq 3$ for the case $\beta_{ij} = \beta_0 > 0 \forall i \neq j$ for $\beta_0 > B(\lambda_i, \beta_0)$ and another case $\lambda_1 = \cdots = \lambda_N, \beta_{ij} = \beta_i, \forall i \neq j$, where some existence and nonexistence results were obtained. We note that their results and methods were different from ours, and—furthermore—that there is no uniqueness result. In particular, the critical cases when $N = 4$ have not been considered in [12]. See also [13] for general discussion of the ground state of such systems. The paper [38] dealt with the subcritical case when $N = 2, 3$ and $\Omega$ is either a bounded regular domain of $\mathbb{R}^n$ or $\Omega = \mathbb{R}^n$ for radial symmetry space, where the positive least energy solution was obtained. In [39], the authors consider the subcritical case when $N = 2, 3$ with $\beta_{ij} = \beta_i$; they get some existence and nonexistence results about the positive ground state.

However, we remark that in $\mathbb{R}^4$, all the cubic nonlinearities and coupling terms in (1.1) are of critical growth in the sense of the Sobolev embedding, which make the problem much more thorny due to the absence of the compact embedding. To the best of our knowledge, there is no general theorem describing the existence, uniqueness and nonexistence of the ground state to the $N$-coupled system in $\mathbb{R}^4$ as $N \geq 3$. Therefore, in the first part of the current paper, we establish some new theorems for the $N$-component solitary wave solutions to the system (1.2) in $\mathbb{R}^4$; hence, the solutions to (1.1). In the second part of the present paper, we re-focus on the case $\dim n = 2, 3$ and $N \geq 3$, and obtain the uniqueness of the ground state for the $N$-coupled nonlinear Schrödinger system (1.1).

We remark that our methods in the present paper are more universal not only for subcritical, critical $N$-coupled systems but also for the much more complicated critical cases—as an example, see Luo and Zou [27] on critical Schrödinger systems involving the Hardy term.

Throughout this paper, we set

$$
B := \begin{pmatrix}
\beta_{11} & \beta_{12} & \cdots & \beta_{1N} \\
\beta_{21} & \beta_{22} & \cdots & \beta_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
\beta_{N1} & \beta_{N2} & \cdots & \beta_{NN}
\end{pmatrix},
$$

where $\beta_{ij} = \beta_i$ for all $i,j = 1, 2, \cdots, N$.

1.1. The critical case: $n = 4$

In this subsection, we consider the existence, uniqueness and nonexistence of the least energy solution to (1.1) when $n = 4$. Firstly, when $\Omega$ is a star-shaped domain and $n = 4$, multiplying the system (1.1) by $(x \cdot \nabla u_j)$ and integrating by parts, we have
\[
\sum_{j=1}^{N} \int_{\Omega} |\nabla u_j|^2 (x \cdot \vec{n}) + (n-2) \sum_{j=1}^{N} \int_{\Omega} |\nabla u_j|^2 + n \sum_{j=1}^{N} \int_{\Omega} \lambda_j u_j^2 = \frac{n}{2} \sum_{1 \leq i < j \leq N} \int_{\Omega} \beta_{ij} u_i^2 u_j^2.
\]

On the other hand, multiplying the system (1.1) by \( u_j \) and integrating by parts, we get
\[
\sum_{j=1}^{N} \int_{\Omega} (|\nabla u_j|^2 + \lambda_j u_j^2) = \sum_{1 \leq i < j \leq N} \int_{\Omega} \beta_{ij} u_i^2 u_j^2.
\]

Combining the two identities above we get that
\[
\sum_{j=1}^{N} \int_{\Omega} |\nabla u_j|^2 (x \cdot \vec{n}) + \left( \frac{N}{2} \sum_{j=1}^{N} \int_{\Omega} u_j^2 \right) = - \frac{n}{2} \sum_{j=1}^{N} \int_{\Omega} |\nabla u_j|^2 = 0. \tag{1.3}
\]

Note that in the last identity above we have used that \( n = 4 \). Therefore, from the Pohozaev type identity (1.3), we know that if \( \Omega \) is a star-shaped domain, and all \( \lambda_j \) are positive, then all \( u_j = 0 \). Therefore, we assume that \( \lambda_j < 0, j = 1, 2, \ldots, n \) when \( \Omega \subset \mathbb{R}^4 \) is a smooth bounded domain.

**Definition 1.1.** We call a solution \((u_1, \ldots, u_N)\) nontrivial if \( u_j \neq 0, 1 \leq j \leq N \); and call \((u_1, \ldots, u_N)\) semi-trivial if there exists some \( i_0, 1 \leq i_0 \leq N \) such that \( u_{i_0} \equiv 0 \). (See, for example, \([1, 4, 24]\)).

1.1. When \( \Omega \) is a smooth bounded domain of \( \mathbb{R}^4 \). We first consider when \( \Omega \) is a smooth bounded domain of \( \mathbb{R}^4 \).

Define \( H := \mathbb{H}^1_0 \times \cdots \times \mathbb{H}^1_0 \). Then the solutions of (1.1) correspond to the critical points of the \( C^2 \) functional \( E : H \rightarrow \mathbb{R} \) given by
\[
E(u_1, \ldots, u_N) = \left\{ \sum_{j=1}^{N} \frac{1}{2} \int_{\Omega} (|\nabla u_j|^2 + \lambda_j u_j^2) - \int_{\Omega} \sum_{1 \leq k \neq j \leq N} \frac{1}{4} \beta_{kj} u_k^2 u_j^2 \right\}. \tag{1.4}
\]

We say a solution \((u_1, \ldots, u_N)\) of (1.1) is a ground state or a least energy solution if \((u_1, \ldots, u_N)\) is nontrivial and \( E(u_1, \ldots, u_N) \leq E(\varphi_1, \ldots, \varphi_N) \) for any other nontrivial solution \((\varphi_1, \ldots, \varphi_N)\) of (1.1). We define
\[
\mathcal{M} := \left\{ (u_1, \ldots, u_N) \in H : u_j \neq 0, \int_{\Omega} (|\nabla u_j|^2 + \lambda_j u_j^2) = \int_{\Omega} \sum_{k \neq j} \beta_{kj} u_k^2 u_j^2, \quad j = 1, \ldots, N \right\}. \tag{1.5}
\]

We can select a sequence of smooth functions \( \phi_j, j = 1, 2, \ldots, N \), which have disjoint compact supports—that is, \( \text{supp}(\phi_j) \cap \text{supp}(\phi_k) = \emptyset \) for \( j \neq k \). Let \( u_j = c_j \phi_j, j = 1, 2, \ldots, N \). Then \( (u_1, u_2, \ldots, u_N) \in \mathcal{M} \) is equivalent to the algebraic equations \( \beta_{ij} c_i^4 \int_{\Omega} \phi_j^2 = c_j^4 \int_{\Omega} (|\nabla \phi_j|^2 + \lambda_j \phi_j^2), j = 1, 2, \ldots, N \), having a solution \((c_1, c_2, \ldots, c_N)\) satisfying \( c_j \neq 0 \) for \( j = 1, 2, \ldots, N \). This can be realized by a proper selection of \( \phi_j \) when \( \beta_{jj} > 0 \) for \( j = 1, 2, \ldots, N \). Hence, \( \mathcal{M} \neq \emptyset \) when \( \beta_{jj} > 0 \) for \( j = 1, 2, \ldots, N \), and all nontrivial solutions of (1.1) belong to \( \mathcal{M} \). We set
Let \( \lambda_1(\Omega) \) be the first eigenvalue of \(-\Delta\) with the Dirichlet boundary condition when \( \Omega \) is a smooth bounded domain. When \(-\lambda_1(\Omega) < \lambda_j < 0, \beta_{jj} > 0\) for \( j = 1, \ldots, N \), we remark that (1.1) has semi-trivial solutions \((0, \ldots, 0, u_{\beta_j}, 0, \ldots, 0)\) for \( 1 \leq j \leq N \), where \( u_{\beta_j} \) is a positive least energy solution of the critical exponent equation
\[
-\Delta u + \lambda_j u = \beta_j u^3, \quad u \geq 0, \quad u \in H_0^1(\Omega),
\] which was first studied by Brezis and Nirenberg in [7]. We will see that the least energy solution of (1.1) is closely related to that of (1.7).

**Theorem 1.1 (Existence).** Suppose that \( \Omega \) is a smooth bounded domain. Assume that \(-\lambda_1(\Omega) < \lambda_1 = \cdots = \lambda_N = \lambda < 0\). If \( \beta_{jj} \geq 0, \beta_{kk} > 0, \forall k, i \neq j \), \( B \) is invertible, and the sum of each column of \( B^{-1} \) is greater than 0, then \( B \) is attained by \( \left( \sqrt{c_1} \omega, \sqrt{c_2} \omega, \ldots, \sqrt{c_N} \omega \right) \), where \( c_j > 0 \) satisfies
\[
\sum_{k=1}^{N} \beta_{jk} c_k = 1, \quad j = 1, \cdots, N.
\]
That is, \( \left( \sqrt{c_1} \omega, \sqrt{c_2} \omega, \ldots, \sqrt{c_N} \omega \right) \) is a positive least energy solution of (1.1), where \( \omega \) is the ground state solution to the following equation
\[
-\Delta u + \lambda u = u^3, \quad u \geq 0, \quad u \in H_0^1(\Omega).
\]

**Remark 1.1.** The ground state solution to the above equation (1.9) depends on dimension; since here \( \Omega \) is a smooth bounded domain of \( \mathbb{R}^d \), it might be clearer if we denoted the solution as \( \omega_\lambda \), to indicate this dependence on dimension; however, to avoid repetition in section 3 below, and for simplicity, we denote it here as \( \omega \).

In theorem 1.1, we know that under some general conditions on the coupling matrix \( B \), we can obtain the existence of least energy solutions; conversely, in theorem 1.2 below, we show that all the least energy solutions must have the following form: every component is some constant multiplying the positive least energy solution of the corresponding scalar equation.

**Theorem 1.2 (Uniqueness).** Suppose that \( \Omega \) is a smooth bounded domain. Assume that \(-\lambda_1(\Omega) < \lambda_1 = \cdots = \lambda_N = \lambda < 0\) and that the matrix \( B \) is positively or negatively definite, \( \beta_{jj} \geq 0, \beta_{kk} > 0, \forall k, i \neq j \). Moreover, suppose that the sum of each column of \( B^{-1} \) is greater than 0. Let \((u_1, u_2, \ldots, u_n)\) be a least energy solution of (1.1), then \((u_1, u_2, \ldots, u_n) = \left( \sqrt{c_1} \omega, \sqrt{c_2} \omega, \ldots, \sqrt{c_N} \omega \right) \), where \( \sum_{k=1}^{N} \beta_{jk} c_k = 1, \quad j = 1, 2, \cdots, N \), and \( \omega \) is a positive least energy solution of \(-\Delta u + \lambda u = u^3, \quad u \geq 0, \quad u \in H_0^1(\Omega) \). In particular, the least energy solution of (1.1) is unique if \( \Omega \) is a ball in \( \mathbb{R}^d \).

**Remark 1.2.** For \( N = 2 \), we note that if \( \beta_{12} = \beta_{21} \in [\min(\beta_{11}, \beta_{22}), \max(\beta_{11}, \beta_{22})] \) and \( \beta_{11} \neq \beta_{22} \), then (1.1) has no positive solutions—see Bartsch and Wang [4].

**Remark 1.3.** If \( B \) is positively or negatively definite with \( \beta_{jj} \geq 0, \forall i \neq j \), \( \beta_{kk} > 0, \forall k \), and assuming that \((\beta_{jj})^{-1} = (a_{ii})\) exists and satisfies \( \sum a_{ij} > 0 \) for \( i = 1, 2, \ldots, N \), then in view of theorems 1.1 and 1.2, we give a characterization of the least energy solutions.

**Remark 1.4.** The conditions in theorems 1.1 and 1.2 are very general. Let us examine \( N = 2 \) for an example. Denote \( \beta := \beta_{12} = \beta_{21} \). Let

(1) The matrix \( B \) is positively or negatively definite, \( \beta_{jj} \geq 0, \beta_{kk} > 0, \forall k, i \neq j \), the sum of each column of \( B^{-1} \) is greater than 0,
(2) \( \beta_i \geq 0, \beta_k > 0, \forall k, i \neq j \) and \( B \) is invertible, the sum of each column of \( B^{-1} \) is greater than 0;

(3) \( 0 \leq \beta < \min\{\beta_{11}, \beta_{22}\}, \beta_{11} > 0, \beta_{22} > 0 \) or \( \beta > \max\{\beta_{11}, \beta_{22}\}, \beta_{11} > 0, \beta_{22} > 0. \)

Then conditions (1) and (2) together are equivalent to condition (3). Combined with remark 1.2, theorems 1.1 and 1.2 give a complete description of the positive least energy solution of (1.1) across the full range of the coupling constant \( \beta > 0. \)

1.1.2. When \( \Omega \) is the whole space \( \mathbb{R}^4. \) Consider the existence of the ground state solution to the following problem defined on the entire space \( \mathbb{R}^4: \)

\[
\begin{align*}
\begin{cases}
-\Delta u_j = \beta_j u_j^3 + \sum_{i \neq j} \beta_{ij} u_i^2 u_j, & x \in \mathbb{R}^4, \\
u_j \in D^{1,2}(\mathbb{R}^4), & j = 1, ..., N,
\end{cases}
\tag{1.10}
\end{align*}
\]

where \( D^{1,2}(\mathbb{R}^4) := \{u \in L^2(\mathbb{R}^4) : |\nabla u| \in L^2(\mathbb{R}^4)\} \) with the norm

\[ ||u||_{D^{1,2}} := \left( \int_{\mathbb{R}^4} |\nabla u|^2 \, dx\right)^{1/2}. \]

Let \( S \) be the sharp constant of the embedding \( D^{1,2}(\mathbb{R}^4) \hookrightarrow L^4(\mathbb{R}^4), \)

\[ \int_{\mathbb{R}^4} |\nabla u|^2 \, dx \geq S \left( \int_{\mathbb{R}^4} |u|^4 \, dx \right)^{1/2}. \] \quad (1.11)

For \( \varepsilon > 0 \) and \( y \in \mathbb{R}^4, \) we consider the Aubin–Talenti instanton

\[ U_{\varepsilon,y} := \frac{2\sqrt{\varepsilon \varepsilon}}{e^2 + |x-y|^2}, \] \quad (1.12)

which satisfies \(-\Delta u = u^3 \) in \( \mathbb{R}^4 \) and \( \int_{\mathbb{R}^4} |\nabla U_{\varepsilon,y}|^2 \, dx = \int_{\mathbb{R}^4} |U_{\varepsilon,y}|^4 \, dx = S^2, \) see [2, 40]. Furthermore, the set \( \{U_{\varepsilon,y} : \varepsilon > 0, y \in \mathbb{R}^4\} \) contains all positive solutions of the equation \(-\Delta u = u^3 \) in \( \mathbb{R}^4. \) Thus, (1.10) has semi-trivial solutions

\[ (0, ..., 0, \mu_k, ..., 0), \; k = 1, ..., N. \]

Let \( D := D^{1,2}(\mathbb{R}^4) \times \cdots \times D^{1,2}(\mathbb{R}^4) \); we define a \( C^2 \) functional \( I : D \to \mathbb{R} \) by

\[ I(u_1, ..., u_N) := \sum_{j=1}^N \left( \frac{1}{2} \int_{\mathbb{R}^4} |\nabla u_j|^2 - \frac{1}{4} \int_{\mathbb{R}^4} \beta_j u_j^4 \right) - \sum_{1 \leq k \neq j \leq N} \frac{1}{4} \int_{\mathbb{R}^4} \beta_{kj} u_k^2 u_j^2. \] \quad (1.13)

Let

\[ \mathcal{N} := \left\{(u_1, ..., u_N) \in D : u_j \neq 0, \int_{\mathbb{R}^4} |\nabla u_j|^2 \right\} \]

\[ = \int_{\mathbb{R}^4} \beta_j u_j^4 + \sum_{k \neq j} \int_{\mathbb{R}^4} \beta_{kj} u_k^2 u_j^2; \; j = 1, ..., N. \] \quad (1.14)

Then \( \mathcal{N} \neq \emptyset, \) and all nontrivial solutions of (1.10) belong to \( \mathcal{N}. \) We set

\[ A := \inf_{\mathcal{N}} I. \] \quad (1.15)
Then,

\[
A = \inf_{(u_1, \ldots, u_N) \in \mathbb{N}} \frac{1}{4} \sum_{j=1}^{N} \int_{\mathbb{R}^4} |\nabla u_j|^2 \, dx.
\]

(1.16)

**Theorem 1.3.**

1. If \( \beta_{ij} < 0, \beta_{kk} > 0, \forall k, i \neq j, 1 \leq i, j \leq N \), then \( A \) is not attained, i.e. the ground state solution of (1.10) does not exist.

2. If \( \beta_{ij} \geq 0, \beta_{kk} > 0, \forall k, i \neq j, 1 \leq i, j \leq N \), the matrix \( B \) is invertible, and the sum of each column of \( B^{-1} \) is greater than 0, then \( A \) is given by

\[
\left( \sqrt{c_1} U_{\varepsilon}, \sqrt{c_2} U_{\varepsilon}, \ldots, \sqrt{c_N} U_{\varepsilon} \right),
\]

where \( \sum_{k=1}^{N} \beta_{kj} c_k = 1 (j = 1, \ldots, N) \). That is, \( \left( \sqrt{c_1} U_{\varepsilon}, \ldots, \sqrt{c_N} U_{\varepsilon} \right) \) is a positive least energy solution of (1.10).

In the above theorem, we deal with the cases that all the interactions are attractive, i.e. \( \beta_{ij} > 0, \forall i \neq j \) or that all the interactions are repulsive, i.e. \( \beta_{ij} \leq 0, \forall i \neq j \); for both cases, we get some existence and nonexistence theorems about the ground state solutions.

However, when attraction and repulsion coexist, i.e. some of \( \beta_{ij}, i \neq j \) are positive but some others are negative, the problem becomes very complicated, and more interesting. Roughly speaking, the next theorem shows that if only parts of states are repulsive to all others, then the ground state solution does not exist. Precisely, we have

**Theorem 1.4.** Assume that \( \beta_{kk} > 0, k = 1, 2, \ldots, N \). Let \( B_1 \) denote the matrix by removing the \( i_0 \)th column and the \( i_0 \)th row of \( B \). Assume that

\[
\beta_{ij} < 0, \forall j \neq i_0 \quad \text{and} \quad \beta_{ij} > 0, \forall i \neq i_0, j \not\in \{i, i_0\}
\]

(1.17)

and that the matrix \( B_1 \) is positively definite. If \( B_1 \) is invertible and the sum of every column of \( B_1^{-1} \) is greater than 0, then the ground state solution to (1.10) does not exist.

**Remark 1.5.** When \( N = 3 \), the hypotheses in theorem 1.4 are satisfied by the following:

\( \beta_{11}, \beta_{22}, \beta_{33} > 0, \beta_{31} < 0, \beta_{32} < 0, 0 < \beta_{12} < \min \{\beta_{11}, \beta_{22}, \beta_{33}\} \).

More generally, by a proof analogous to that of theorem 1.4, we have the following theorem.

**Theorem 1.5.** Assume that \( \beta_{kk} > 0, k = 1, 2, \ldots, N \). Let \( B_1 \) denote the matrix constructed by removing the \( i_1, \ldots, i_k \)th columns and the \( i_1, \ldots, i_k \)th rows of \( B \). Suppose that \( \beta_{ij} \) satisfies

\[
\beta_{ij} < 0, \quad \text{for} \quad i = i_1, \ldots, i_k, \quad j \neq i_1, \ldots, i_k; \quad \beta_{ij} > 0, \quad \text{for} \quad i \not\in \{i_1, \ldots, i_k\}, \quad j \neq i.
\]

and that the matrix \( B_1 \) is positively definite. If, further, the sum of every column of \( B_1^{-1} \) is greater than 0, then the ground state solution to (1.10) does not exist.

1.2. The subcritical case in \( \mathbb{R}^n \), i.e. \( n = 2, 3 \)

In this subsection, we are going to study the existence, uniqueness and radial symmetry of the following systems defined in the entire space \( \mathbb{R}^n \) for \( n = 2, 3 \):
\[
\begin{aligned}
\left\{ -\Delta u_j + \lambda u_j = \beta_j u_j^3 + \sum_{k \neq j} \beta_k u_k^2 u_j, \quad x \in \mathbb{R}^n; \right. \\
\left. u_j \geq 0, x \in \mathbb{R}^n; \quad u(x) \to 0 \text{ as } |x| \to +\infty; \quad j = 1, \ldots, N. \right. 
\end{aligned}
\]  
(1.18)

We focus on the case of \( N \geq 3 \). Let \( \omega_1(x) = \omega_1(|x|) \) be the unique positive solution of the equation

\[
-\Delta \omega + \omega = \omega^3, \quad x \in \mathbb{R}^n.
\]  
(1.19)

By [17] we see that any positive solution of (1.19) is radially symmetric and strictly decreasing in the radial variable. The uniqueness of the radial solution of (1.19) is due to [11] (see also [23]). Consider the minimization problem:

\[
S_{\lambda, \mu} = \inf_{u \in H^1(\mathbb{R}^n) \setminus \{0\}} \frac{\|u\|_2^2}{\|\mu u\|_2^2} \quad \text{and} \quad T_{\lambda, \mu} = \inf_{u \in \mathcal{N}_0} \left\{ \frac{1}{2} \|u\|_2^2 - \frac{1}{4} \int_{\mathbb{R}^n} \mu u^4 \right\},
\]  
(1.20)

where \( \|u\|_2^2 := \int_{\mathbb{R}^n} (|\nabla u|^2 + \lambda u^2) \) and \( \mathcal{N}_0 := \{ u \in H^1(\mathbb{R}^n) : u \neq 0, \|u\|_2^2 = \int_{\mathbb{R}^n} \mu u^4 \} \). It is well known (see e.g. [44]) that the function \( \omega_{\lambda, \mu}(x) := \mu^{-\frac{1}{4}} \sqrt{\lambda} \omega_1(\sqrt{\lambda} x) \) is a minimizer for \( T_{\lambda, \mu} \), and is also the unique solution to the equation

\[
-\Delta \omega + \lambda \omega = \mu \omega^3 \quad \text{in} \quad \mathbb{R}^n.
\]  
(1.21)

In addition, \( T_{\lambda, \mu} = \frac{1}{4} S_{\lambda, \mu}^2, S_{\lambda, \mu} = \mu^{-\frac{1}{4}} \lambda^{-\frac{1}{4}} S_{1,1} \). It follows that

\[
\int_{\mathbb{R}^n} (|\nabla u|^2 + \lambda u^2) \, dx \geq 2 \sqrt{T_{1,1}} \left( \int_{\mathbb{R}^n} u^4 \, dx \right)^{1/2}, \quad u \in H^1(\mathbb{R}^n).
\]  
(1.22)

Let \( D := D^{1,2}(\mathbb{R}^n) \times \cdots \times D^{1,2}(\mathbb{R}^n) \); we define a \( C^2 \) functional \( I : D \to \mathbb{R} \) by

\[
I(u_1, \ldots, u_N) = \frac{1}{2} \sum_{j=1}^N \int_{\mathbb{R}^n} (|\nabla u_j|^2 + \lambda u_j^2) - \frac{1}{4} \int_{\mathbb{R}^n} \beta_j u_j^4
\]

\[
- \sum_{1 \leq k \neq j \leq N} \frac{1}{4} \int_{\mathbb{R}^n} \beta_k u_k^2 u_j^2.
\]  
(1.23)

Set

\[
\mathcal{N} = \left\{ (u_1, \ldots, u_N) \in D : \quad u_j \neq 0, \quad \int_{\mathbb{R}^n} (|\nabla u_j|^2 + \lambda u_j^2) \right. \\
\left. = \int_{\mathbb{R}^n} \beta_j u_j^4 + \sum_{k \neq j} \int_{\mathbb{R}^n} \beta_k u_k^2 u_j^2; \quad j = 1, \ldots, N \right\}.
\]  
(1.24)

Then \( \mathcal{N} \neq \emptyset \), and all nontrivial solutions of (1.18) belong to \( \mathcal{N} \). We set

\[
T := \inf_{\mathcal{N}} I
\]  
(1.25)

and denote the inverse matrix of \( B \) by \( B^{-1} = (\beta^{ij}) \), if it exists.

**Theorem 1.6 (Existence).** Assume that \( \lambda > 0 \) and \( \det B \neq 0 \), \( \beta_{ij} \geq 0, \beta_{kk} > 0, \forall k, i \neq j \). If \( \sum_{k=1}^N \beta^{ij} c_k > 0 \) for all \( j = 1, 2, \ldots, N \), then \( T \) is attained by

\[
(\sqrt{c_1} \omega_{1,1}, \sqrt{c_1} \omega_{1,1}, \cdots, \sqrt{c_N} \omega_{1,1}),
\]

where \( c_j > 0 \) satisfies \( \sum_{k=1}^N \beta_{jk} c_k = 1 \) for all \( j = 1, 2, \cdots, N \). That is,

\[
(\sqrt{c_1} \omega_{1,1}, \sqrt{c_1} \omega_{1,1}, \cdots, \sqrt{c_N} \omega_{1,1})
\]

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is a positive least energy solution of (1.18).

**Theorem 1.7 (Uniqueness).** Assume that $\lambda > 0$ and the matrix $B$ is positively or negatively definite, $\beta_j > 0$, $\beta_{ik} > 0$, $\forall k, i \neq j$. Suppose that $\sum_{k=1}^{N} \beta_{ki} > 0$ for all $j = 1, 2, \cdots, N$. Let $(u_1, u_2, \ldots, u_n)$ be a least energy solution of (1.18), then

$$
(u_1, u_2, \ldots, u_n) = \left( \sqrt{\beta_{11}} \omega_{1}, \sqrt{\beta_{22}} \omega_{2}, \cdots, \sqrt{\beta_{nn}} \omega_{n} \right),
$$

where $c_j$ satisfies $\sum_{j=1}^{N} \beta_{kj} c_k = 1 (j = 1, 2, \cdots, N)$ and $\omega_{kj}$ is the unique positive (also radially symmetric) solution of $-\Delta u + \lambda u = u^3$ in $\mathbb{R}^n$.

The paper is organized as follows. In section 2, we first prepare some lemmas which hold for general spatial dimension $n$. In section 3, we deal with the critical cases when $n = 4$. Section 4 will be devoted to the subcritical case, i.e. $n = 2, 3$.

## 2. Preliminaries

In this section, we present some lemmas which will be used several times throughout the paper. We remark that the notations in this section are self-contained and independent of other sections.

The following lemmas provide a unified tool for the proof of the uniqueness when the general $N$-coupled system is concerned.

### Lemma 2.1

Assume $P = (p_{ij})$ is a real symmetric matrix, $P^* = (P^*_{ij})$ is the adjoint matrix of $P$. Then for any $k, l \in \{1, 2, \ldots, N\}$, we have

$$
\frac{\partial P^*_{jl}}{\partial p_{ki}} \det(P) = P^*_{jl} P^*_{ki} - P^*_{kl} P^*_{ji}.
$$

(2.1)

**Proof.** We first assume that the matrix $P$ is invertible. For the simplicity, we just prove the case of $k = l = 1$. Since $P^* P = \det(P) I$, that is,

$$
\sum_{k=1}^{N} P^*_{jk} p_{kj} = \det(P) \delta_{ij}, \quad i, j = 1, \cdots, N,
$$

(2.2)

where $\delta_{ij}$ is the Kronecker notation. Derivative both sides of (2.2), we obtain

$$
\sum_{k=1}^{N} \left( \frac{\partial P^*_{jl}}{\partial p_{kl}} p_{kj} + P^*_{jk} \frac{\partial p_{kl}}{\partial p_{kl}} \right) = \frac{\partial \det(P)}{\partial p_{kl}} \delta_{ij},
$$

and the left-hand side is equal to $\sum_{k=1}^{N} \left( \frac{\partial P^*_{jl}}{\partial p_{kl}} p_{kj} + P^*_{lk} \delta_{ik} \delta_{lj} \right)$. Therefore,

$$
\sum_{k=1}^{N} \frac{\partial P^*_{jl}}{\partial p_{kl}} a_{kj} = P^*_{jl} \delta_{ij} - P^*_{lk} \delta_{lj} = \frac{\partial P^*_{jk}}{\partial a_{kl}} = \sum_{j=1}^{N} (P^*_{jk} \delta_{il} - P^*_{lj} \delta_{ik}) \frac{P^*_{jl}}{\det(P)}.
$$

(Here we have used the fact that $\frac{\partial \det(P)}{\partial p_{kl}} = P^*_{kl}$, since $\det(P) = p_{11} P^*_{11} + p_{12} P^*_{21} + \cdots + p_{1N} P^*_{N1}$).

That is, $\frac{\partial P^*_{jl}}{\partial p_{kl}} \det(P) = P^*_{jl} P^*_{kl} - P_{lk} P^*_{jl}$.

If matrix $P$ is not invertible, i.e. $\det(P) = 0$, then we can select a sequence of $\varepsilon_k \to 0$ such that $\det(P + \varepsilon_k I) \neq 0$, where $I$ is the identity matrix, then we can apply the above result, and
let $\varepsilon_k \to 0$, to get the desired conclusion.

**Lemma 2.2.** Assume $\det(B) \neq 0$, and let $B^* = (B_{ij}^*)$ be the adjoint of $B$. For any fixed $\beta_{ml}$, we regard $\beta_{ml}$ as one variable; all the others $\beta_{ij}$ for $i \neq m$ or $j \neq l$ as constants. We consider the following one variable function $W(\beta_{ml}) := \sum_{j=1}^{N} c_j$, where $c_j$ is determined by the equation $\sum_{i=1}^{N} \beta_{ij} c_i = 1, j = 1, 2, \ldots, N$. Then $W'(\beta_{ml}) = \frac{\partial W(\beta_{ml})}{\partial \beta_{ml}} = -c_m c_l$.

**Proof.** Since $c_j = \sum_{k=1}^{N} \frac{\beta_{jk}^*}{\det(B)}$, we have $W(\beta_{ml}) = \sum_{j,k=1}^{N} \frac{\beta_{jk}^*}{\det(B)}$. Hence,

$$W'(\beta_{ml}) = \sum_{j,k=1}^{N} \frac{1}{(\det(B))^2} \left( \frac{\partial \beta_{jk}^*}{\partial \beta_{ml}} \det(B) - \beta_{jk} \frac{\partial \det(B)}{\partial \beta_{ml}} \right).$$

By lemma 2.1, we obtain that

$$W'(\beta_{ml}) = -\sum_{j,k=1}^{N} \frac{1}{(\det(B))^2} \beta_{jk} \beta_{mk}.$$

Thus, $W'(\beta_{ml}) = -c_m c_l$. The conclusion follows.

The following result (1) is probably known, but we do not have the proper reference on hand; result (2) will usually be used in our proof, and has also appeared in [24]; we give brief proofs here for the reader’s convenience.

**Lemma 2.3.**

1. If $A = (a_{ij})_{N \times N}$ is a strict diagonal dominant matrix, and the diagonal elements are positive, i.e. $a_{ii} - \sum_{j \neq i} |a_{ij}| > 0$, $a_{ii} > 0$, $i = 1, \ldots, N$, then all eigenvalues of $A$ are positive. In particular, if further, $A$ is symmetric, then $A$ is positively definite and $\det(A) \geq \left( \min_{1 \leq i \leq N} (a_{ii} - \sum_{j \neq i} |a_{ij}|) \right)^N$.

2. If $u_i(x) \neq 0$ in $\Omega$ for $j = 1, 2, \ldots, N$ and the matrix $\beta = (\beta_{ij})$ is positively definite (or negatively definite), then the matrix $(\beta_{ij} \int_{\Omega} u_i^2 u_j^2)$ is positively definite (resp. negatively definite).

**Proof.** We first prove (1). For any eigenvalue $\lambda$ of $A$, let $x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N$ be the corresponding eigenvector. Let $x_0$ be the absolutely maximal element, that is $|x_0| = \max_{1 \leq i \leq N} |x_i|$ (obviously $x_0 \neq 0$), then normalize the eigenvector $x = (x_1, x_2, \ldots, x_N)$ by multiplying the constant $\frac{1}{\sqrt{|x_0|}}$. In this way, we may assume $|x_j| \leq 1$ for $1 \leq j \leq N$ and $x_0 = 1$. Then $a_{00} + \sum_{j \neq 0} |a_{0j}| x_j = \lambda$. By re-scaling of the eigenvector, we obtain that $\lambda \geq a_{00} - \sum_{j \neq 0} |a_{0j}| x_j > 0$. Recall that $\det(A) = \prod_{j=1}^{N} \lambda_j$; the conclusion is proved.

Now we turn to prove (2); we only prove the case when $\beta = (\beta_{ij})$ is positively definite—the other case is similar. Let $y = (y_i) \in \mathbb{R}^N$, since the matrix $\beta = (\beta_{ij})$ is positively definite, we observe that

$$\sum_{ij} \left( \int_{\Omega} \beta_{ij} u_i^2 u_j^2 \, dx \right) y_i y_j = \int_{\Omega} \sum_{ij} \left( \beta_{ij} (u_i^2 y_i) (u_j^2 y_j) \right) \, dx \geq 0, \quad (2.3)$$

and the equality holds if and only if for almost everywhere $x \in \Omega$,

$$(u_1^2(x)y_1, u_2^2(x)y_2, \ldots, u_N^2(x)y_N) = (0, 0, \ldots, 0),$$

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since \( u_i(x) \neq 0 \) in \( \Omega \) for \( j = 1, 2, \ldots, N \), which implies that \( y = (y_1, y_2, \ldots, y_N) \) is the vector \( \mathbf{0} \). Therefore, \( (\beta_j \int_{\Omega} u_j^2 u_i^2) \) is positively definite.

The following lemma is one of the key points to prove theorems 1.1, 1.3(2) and 1.6. Essentially, this lemma is an indispensable part in the proof of the existence of the ground state solutions for the \( N \)-coupled system.

**Lemma 2.4.** Let \( \tilde{A} = (a_{ij}) \) and \( \tilde{B} = (b_{ij}) \) denote the \( N \times N \) matrices and \( \tilde{B} = \tilde{A}^{-1} \). Assume 
\[
\det(\tilde{A}) \neq 0 \quad \text{and} \quad \sum_{k=1}^{N} b_{kj} > 0 \quad \text{for all} \quad j = 1, \ldots, N.
\]
For the sequence \( \varepsilon_m \geq 0 \), suppose that \( x_{i,m} \) satisfying
\[
\begin{align*}
\sum_{i=1}^{N} x_{i,m} &\leq \varepsilon_m, \\
\sum_{i=1}^{N} a_{ij} x_{j,m} &\geq 0, \quad k = 1, \ldots, N.
\end{align*}
\]
Then \( 0 \leq \sum_{i=1}^{N} x_{i,m} \leq \varepsilon_m \). Furthermore, if \( \varepsilon_m \to 0 \) as \( m \to \infty \), then \( x_{i,m} \to 0 \) as \( m \to \infty \) for each \( k = 1, \ldots, N \).

**Proof.** Denote \( \theta_{k,m} := \sum_{j=1}^{N} a_{kj} x_{j,m} \) for all \( k = 1, \ldots, N \). Then \( \theta_{k,m} \geq 0 \) by (2.4). Further, we have \( x_{k,m} = \sum_{j=1}^{N} b_{kj} \theta_{j,m} \). Thus,
\[
\sum_{i=1}^{N} x_{i,m} = \sum_{k=1}^{N} \sum_{j=1}^{N} b_{kj} \theta_{j,m} = \sum_{j=1}^{N} \theta_{j,m} \left( \sum_{k=1}^{N} b_{kj} \right)
\]
Since \( \sum_{k=1}^{N} b_{kj} > 0 \), we have \( \sum_{k=1}^{N} x_{k,m} = \sum_{i=1}^{N} \theta_{i,m} \left( \sum_{k=1}^{N} b_{kj} \right) \geq 0 \). Combining this with the first inequality in (2.4), we obtain that \( 0 \leq \sum_{k=1}^{N} x_{k,m} \leq \varepsilon_m \to 0 \) as \( m \to \infty \) for each \( j = 1, \ldots, N \). It follows that \( x_{j,m} \to 0 \) as \( m \to \infty \) for any \( k = 1, \ldots, N \).

Let \( A \) and \( B \) be defined as in (1.15) and (1.6), \( M \) and \( N \) be defined as in (1.5) and (1.24), \( I \) and \( E \) be defined as in (1.4) and (1.24), respectively. Next, we show that the minimizers of \( A \) and \( B \) on the Nehari manifolds are in fact the global critical points of the corresponding functionals.

**Lemma 2.5.** Assume \( A \) (or \( B \)) is attained by \( u \in N \) (resp. \( u \in M \)). Then \( u \) is a critical point of \( I \) (resp. \( E \)) provided that either \( \beta_i < 0 \) for all \( i \neq j (1 \leq i, j \leq N) \) or \( B \) is positively definite.

**Proof.** We just prove the conclusion for \( A \). Let \( u = (u_1, u_2, \ldots, u_N) \). Recall that
\[
I(u) = \frac{1}{2} \sum_{i=1}^{N} \int_{\Omega} |\nabla u_i|^2 - \frac{1}{2} \sum_{i=1}^{N} \int_{\Omega} \beta_i u_i^4 - \frac{1}{4} \sum_{1 \leq i \neq j \leq N} \int_{\Omega} \beta_{ij} u_i^2 u_j^2.
\]
Set
\[
G_i(u) = \int_{\Omega} |\nabla u_i|^2 - \beta_i \int_{\Omega} u_i^4 - \sum_{j \neq i} \int_{\Omega} \beta_{ij} u_i^2 u_j^2, \quad i = 1, \ldots, N.
\]
Then \( N = \{ u = (u_1, u_2, \ldots, u_N) : u_i \neq 0, G_i(u) = 0, \text{ for } i = 1, 2, \ldots, N \} \). Assume \( A \) is attained by \( u \in N \), then there are Lagrange multipliers \( L_i \) such that
\[
I'(u) + \sum_{i=1}^{N} L_i G_i'(u) = 0. \tag{2.6}
\]
Let $\phi_j = (0, \ldots, 0, u_j, 0, \ldots, 0)$. Since $u \in N$ we obtain $\sum_{i=1}^{N} L_i(G'(u), \phi_j) = 0$ for all $j$, where $(\cdot, \cdot)$ denote the action between the dual space and the original space. Which is equivalent to $\sum_{i=1}^{N} \left( \int_{\Omega} \beta_i u_i^2 u_j^2 \right) L_i = 0$ for all $j$. By lemma 2.3, for both cases we know that the matrix $P := \left( \beta_i \int_{\Omega} u_i^2 u_j^2 \right)$ is positively definite, which implies that $L_i = 0$ for all $i = 1, \ldots, N$. Hence, $I'(u) = 0$, because of (2.6).

3. The critical case: $n = 4$

Firstly, we recall (see [7]) that the following problem

$$-\Delta u + \lambda u = u^3, \quad u \geq 0, \quad u \in H^1_0(\Omega),$$

has a positive least energy solution $\omega$ with energy

$$B_1 := \frac{1}{4} \int_{\Omega} |\nabla \omega|^2 + \lambda \omega^2 = \frac{1}{4} \int_{\Omega} \omega^4,$$

where $-\lambda_1(\Omega) < \lambda < 0$. Moreover, we have

$$\int_{\Omega} (|\nabla u|^2 + \lambda u^2) dx \geq 2 \sqrt{B_1} \left( \int_{\Omega} u^4 dx \right)^{1/2}, \quad u \in H^1_0(\Omega).$$

3.1. Proof of theorem 1.1 (existence), where $\Omega$ is a smooth bounded domain, and $n = 4$

Recalling that $B$ is invertible, and the sum of each row of $B^{-1}$ is greater than 0, it follows that equation (1.8) has a solution $(c_1, \ldots, c_N)$ satisfying $c_j > 0$ for all $1 \leq j \leq N$. Recall that $\omega$ is the positive least energy solution to (3.1); we see that $(\sqrt{c_1} \omega, \sqrt{c_2} \omega, \ldots, \sqrt{c_N} \omega)$ is a nontrivial solution of (1.1), and

$$E(\sqrt{c_1} \omega, \ldots, \sqrt{c_N} \omega) = \sum_{j=1}^{N} c_j B_1.$$  \hspace{1cm} (3.4)

Let $\{(u_{1,m}, \ldots, u_{N,m})\}_{m=1}^{\infty} \subset M$ be a minimizing sequence for $B$, that is,

$$E(u_{1,m}, \ldots, u_{N,m}) \to B.$$  \hspace{1cm} (3.5)

Define $d_{jm} = ( \int_{\Omega} u_{jm}^4 dx )^{1/2}$ for $i = 1, \ldots, N$. It follows from (3.3) that we have

$$2 \sqrt{B_1} d_{jm} \leq \int_{\Omega} (|\nabla u_{jm}|^2 + \lambda u_{jm}^2)$$

$$= \int_{\Omega} \beta_j u_{jm}^4 + \sum_{k \neq j} \int_{\Omega} \beta_{kj} u_{km}^2 u_{jm}^2$$

$$\leq \beta_j d_{jm}^2 + \sum_{k \neq j} \beta_{kj} d_{jm} d_{km}. $$ \hspace{1cm} (3.6)

On the other hand,

$$2 \sqrt{B_1} \sum_{i=1}^{N} d_{jm} \leq 4 E(u_{1,m}, \ldots, u_{N,m}) \leq 4 \sum_{j=1}^{N} c_j B_1 + o(1). $$ \hspace{1cm} (3.7)
Thus we have
\[ \begin{align*}
\sum_{i=1}^{N} d_{i,m} & \leq \sum_{i=1}^{N} c_i 2 \sqrt{B_i} + o(1), \\
2 \sqrt{B_i} & \leq \beta_i d_{i,m} + \sum_{k \neq i} \beta_k d_{k,m}.
\end{align*} \tag{3.8} \]
Recall condition (1.8), then the inequalities above are equivalent to
\[ \begin{align*}
\sum_{i=1}^{N} (d_{i,m} - c_i 2 \sqrt{B_i}) & \leq o(1), \\
\beta_i (d_{i,m} - c_i 2 \sqrt{B_i}) + \sum_{k \neq i} \beta_k (d_{k,m} - c_k 2 \sqrt{B_k}) & \geq 0,
\end{align*} \tag{3.9} \]
By lemma 2.4, we have \( d_{i,m} \to c_i 2 \sqrt{B_i} \) as \( m \to \infty \), and
\[ 4B = \lim_{m \to \infty} 4E(u_{1,m}, \ldots, u_{N,m}) \geq \lim_{m \to \infty} 2 \sqrt{B_i} \sum_{i=1}^{N} d_{i,m} = 4 \sum_{i=1}^{N} c_i B_i. \tag{3.10} \]
Combining this with (3.4), one has that \( B = \sum_{i=1}^{N} c_i B_i = E(\sqrt{c_i} \omega, \ldots, \sqrt{c_n} \omega) \), which implies that \((\sqrt{c_1} \omega, \ldots, \sqrt{c_n} \omega)\) is a positive least energy solution of (1.1).

3.2. Proof of theorem 1.2 (uniqueness), where \( \Omega \) is a smooth bounded domain and \( n = 4 \)

**Proof.** Let \((u_{1,0}, \ldots, u_{N,0})\) be a least energy solution of (1.1), hence \( u_{j,0} \geq 0 \) for \( j = 1, 2, \ldots, N \); we rewrite the equation by \(-\Delta u_j + \sum_{i \neq j} \beta^+_{ij} u_i^2 u_j = (-\lambda_j + \beta^-_{ij}) u_j + \sum_{i \neq j} \beta^-_{ij} u_i^2 u_j \geq 0\), where \( \beta^+_{ij} \) and \( \beta^-_{ij} \) denote the positive and negative parts of \( \beta_{ij} \), respectively. Then \( u_{j,0} \geq 0 \) for all \( j = 1, \ldots, N \) by the strong maximum principle. More precisely, throughout this subsection, for each fixed pair \((m, l)\) with \( 1 \leq m, l \leq N \), we view \( \beta_{ml} \) as a variable. Recalling the definitions of \( E, M \) and \( B \), they all become the \( \beta_{ml} \)-dependent functions. We use notations \( E_{\beta_{ml}}, M_{\beta_{ml}} \) and \( B(\beta_{ml}) \). For each \( j \), we define the following function of \((t_1, \ldots, t_N)\):
\[ f_j(t_1, \ldots, t_N) = \int_{\Omega} t_j \beta_j u_j^2 \ dx + \sum_{k \neq j} \int_{\Omega} t_k \beta_{kj} u_k^2 u_j^2 - \int_{\Omega} ||\nabla u_j||^2 + \lambda u_j^2. \]
Hence, we have \( f_j(1, \ldots, 1) = 0 \) and \( \frac{\partial f_j}{\partial t_i} = \beta_{ji} \int_{\Omega} u_i^2 u_j^2. \) Let the matrix
\[ F := \left( \frac{\partial f_j}{\partial t_i} \right). \tag{3.11} \]
Since \( F \) is positively or negatively definite, so is \( F \) by (2) of lemma 2.3; hence, \( \det(F) \neq 0 \). Therefore, by the implicit function theorem, the functions \( t_j(\beta_{ml}) \) are well-defined for \( \beta_{ml} \in (\beta_{ml} - \delta_1, \beta_{ml} + \delta_1) \), and are of class \( C^1 \) for some \( 0 < \delta_1 < \delta \). Moreover, \( t_j(\beta_{ml}) = 1 \) for all \( j = 1, \ldots, N \). Therefore, we may assume that \( t_j(\beta_{ml}) > 0 \) for all \( \beta_{ml} \in (\beta_{ml} - \delta_1, \beta_{ml} + \delta_1) \) by choosing a small value of \( \delta_1 \). Since \( f_k(t_1(\beta_{ml}), \ldots, t_N(\beta_{ml}), \beta_{ml}) = 0 \) for all \( k = 1, \ldots, N \), a direct computation shows that
\[ \sum_{j=1}^{N} \frac{\partial f_k}{\partial t_j}(\beta_{ml}) = - \frac{\partial f_k}{\partial \beta_{ml}}, \quad t_j'(\beta_{ml}) = - \sum_{k=1}^{N} \frac{\partial f_k}{\partial \beta_{ml}} \frac{F^*}{\det(F)}. \]
here $F^* := (F^*_j)$ is the adjoint matrix of $F$. On the other hand, since
\[
\frac{\partial f_k}{\partial \beta_{ml}} = \delta_{km} \int_{\Omega} u_{m0}^2 u_{l0}^2 \, dx,
\]
it follows that
\[
t_j^* (\beta_{ml}) = \int_{\Omega} u_{m0}^2 u_{l0}^2 \, dx \frac{F^*_{mj}}{\det(F)}. \tag{3.12}
\]
By the Taylor expansion, we see that
\[
t_j (\beta_{ml}) = 1 + t_j^* (\beta_{ml}) (\beta_{ml} - \beta_{ml}) + O((\beta_{ml} - \beta_{ml})^2).
\]
Note that $f_j (t_1 (\beta_{ml}), \ldots, t_N (\beta_{ml}), \beta_{ml}) \equiv 0$ $(j = 1, \ldots, N)$ implies that
\[
\left( \sqrt{t_1 (\beta_{ml})} u_{10}, \ldots, \sqrt{t_N (\beta_{ml})} u_{N0} \right) \in \mathcal{M}_{\beta_{ml}}.
\]
Therefore,
\[
B (\beta_{ml}) \leq E_{\beta_{ml}} \left( \sqrt{t_1 (\beta_{ml})} u_{10}, \ldots, \sqrt{t_N (\beta_{ml})} u_{N0} \right)
\]
\[
= \frac{1}{4} \sum_{j=1}^{N} t_j (\beta_{ml}) \int_{\Omega} (|\nabla u_{j0}|^2 + \lambda u_{j0}^2) \, dx
\]
\[
= B (\beta_{ml}) + \frac{1}{4} D (\beta_{ml} - \beta_{ml}) + O((\beta_{ml} - \beta_{ml})^2), \tag{3.13}
\]
where the constant $D$ is given by
\[
D := \sum_{j=1}^{N} t_j (\beta_{ml}) \int_{\Omega} (|\nabla u_{j0}|^2 + \lambda u_{j0}^2) \, dx
\]
\[
= \sum_{j=1}^{N} t_j (\beta_{ml}) \left( \int_{\Omega} |\beta_{j0} u_{j0}|^2 \, dx + \sum_{k \neq j} \beta_{kj} \int_{\Omega} u_{k0}^2 u_{l0}^2 \, dx \right)
\]
\[
= \int_{\Omega} u_{m0}^2 u_{l0}^2 \sum_{j=1}^{N} \frac{F^*_{mj}}{\det(F)} \left( \sum_{k} F_{kj} \right)
\]
\[
= \int_{\Omega} u_{m0}^2 u_{l0}^2 \frac{1}{\det(F)} \sum_{k=1}^{N} \sum_{j=1}^{N} F^*_{mj} F_{kj}
\]
\[
= \int_{\Omega} u_{m0}^2 u_{l0}^2 \frac{1}{\det(F)} \sum_{k=1}^{N} \sum_{j=1}^{N} \delta_{km} \det(F)
\]
\[
= \int_{\Omega} u_{m0}^2 u_{l0}^2. \tag{3.14}
\]
It follows that
\[
\frac{B (\beta_{ml}) - B (\beta_{ml})}{\beta_{ml} - \beta_{ml}} \geq \frac{D}{4} + O((\beta_{ml} - \beta_{ml})^2)
\]
as \( \tilde{\beta}_{ml} \neq \beta_{ml} \) and so \( B'(\beta_{ml}) \geq \frac{D}{4} \). Similarly, we have \( \frac{B(\beta_{ml}) - B(\beta_{ml})}{\beta_{ml} - \beta_{ml}} \leq \frac{D}{4} + O(\beta_{ml} - \beta_{ml}) \) as \( \beta_{ml} \searrow \beta_{ml} \), i.e. \( B'(\beta_{ml}) \leq \frac{D}{4} \). Hence,

\[
B'(\beta_{ml}) = -\frac{D}{4} = -\frac{1}{4} \int_{\Omega} u_{m0}^2 u_{m0}^2.
\]

On the other hand, by lemma 2.2 and theorem 1.1 (existence of the least energy solution), we have \( B'(\beta_{ml}) = -c_m c_l B_1 = -\frac{1}{4} c_m c_l \int_{\Omega} \omega^4 \). Hence, \( \int_{\Omega} u_{m0}^2 u_{m0}^2 = c_m c_l \int_{\Omega} \omega^4 \). Let

\[
(\tilde{u}_{1,0}, \ldots, \tilde{u}_{N,0}) := \left( \frac{1}{\sqrt{c_1}} u_{1,0}, \ldots, \frac{1}{\sqrt{c_N}} u_{N,0} \right).
\]

(3.15)

Noting that \( \Sigma_j \beta_j c_k = 1 \) for all \( j = 1, \ldots, N \), and that \( (u_{1,0}, \ldots, u_{N,0}) \in \mathcal{M} \), we get

\[
\int_{\Omega} (|\nabla \tilde{u}_{j,0}|^2 + \lambda \tilde{u}_{j,0}^2) dx = \frac{1}{c_j} \int_{\Omega} (|\nabla u_{j,0}|^2 + \lambda u_{j,0}^2) dx
\]

\[
= \frac{1}{c_j} (\int_{\Omega} \beta_j u_{j,0}^4 dx + \sum_{k \neq j} \beta_{kj} \int_{\Omega} u_{k,0}^2 u_{j,0}^2 dx)
\]

\[
= \frac{1}{c_j} (\beta_j c_j + \sum_{k \neq j} \beta_{kj} c_k) \int_{\Omega} \omega^4 dx
\]

\[
= (\beta_j c_j + \sum_{k \neq j} \beta_{kj} c_k) \int_{\Omega} \omega^4 dx = \int_{\Omega} \tilde{u}_{j,0}^4 dx.
\]

(3.16)

Then by (3.3), we have \( \frac{1}{4} \int_{\Omega} (|\nabla \tilde{u}_{j,0}|^2 + \lambda \tilde{u}_{j,0}^2) dx \geq B_1 \) for all \( j = 1, \ldots, N \). Hence,

\[
B = \sum_{j=1}^{N} c_j B_1 = \frac{1}{4} \sum_{j=1}^{N} \int_{\Omega} (|\nabla u_{j,0}|^2 + \lambda u_{j,0}^2) dx
\]

\[
= \frac{1}{4} \sum_{j=1}^{N} c_j \int_{\Omega} (|\nabla \tilde{u}_{j,0}|^2 + \lambda \tilde{u}_{j,0}^2) dx \geq \sum_{j=1}^{N} c_j B_1.
\]

This implies that \( \frac{1}{4} \int_{\Omega} (|\nabla \tilde{u}_{j,0}|^2 + \lambda \tilde{u}_{j,0}^2) dx = B_1, j = 1, \ldots, N \). Combining this with (3.16), we see that \( \tilde{u}_{j,0}(j = 1, \ldots, N) \) are the positive least energy solutions of (1.7). We see that

\[
-\Delta \tilde{u}_{j,0} + \lambda \tilde{u}_{j,0} = \beta_j c_j \tilde{u}_{j,0}^3 + \sum_{k \neq j} \beta_{kj} c_k \tilde{u}_{k,0}^2 \tilde{u}_{j,0} = \tilde{u}_{j,0}^3
\]

\[
\beta_j c_j \tilde{u}_{j,0}^3 + \sum_{k \neq j} \beta_{kj} c_k \tilde{u}_{k,0}^2 = \tilde{u}_{j,0}^3.
\]

Hence, \( \sum_{k \neq j} \beta_{kj} c_k \tilde{u}_{k,0}^2 = 1 \). Since \( B \) is invertible, we obtain that \( \frac{\tilde{u}_{k,0}^2}{\tilde{u}_{j,0}^3} = 1 \) for all \( k \neq j \). Denote \( U = \tilde{u}_{1,0} \), then \( (u_{1,0}, \ldots, u_{N,0}) = (\sqrt{c_1} U, \ldots, \sqrt{c_N} U) \), where \( U \) is a positive least energy solution of (1.7). Finally, when \( \Omega \) is a ball in \( \mathbb{R}^4 \), we know that the least energy solution of the equation (1.7) is unique (see [7]). Therefore, the least energy solution of (1.1) is unique. \( \square \)
3.3. Proof of theorem 1.3 (existence and nonexistence) when the interactions are either all attractive or all repulsive, where \( \Omega = \mathbb{R}^d \) and \( \lambda_j = 0, j = 1, 2, \ldots, N \)

**Proof.** We first prove part (1) of theorem 1.3. Let \( e_i = (1, 0, \ldots, 0) \) and
\[
  u_{j,R}(x) = \omega_j(x + jRe_1), \quad j = 1, \ldots, N,
\]
where \( \omega_j := \beta_j^{-1/2}U_{1,0} \), \( U_{1,0} \) is defined in (1.12), then \( u_{j,R} \to 0 \) weakly in \( D^{1,2}(\mathbb{R}^d) \) and so \( u_{j,R} \to 0 \) weakly in \( L^2(\mathbb{R}^d) \) as \( R \to +\infty \). Without loss of generality, we assume that \( j > i \), thus we get
\[
  \lim_{R \to +\infty} \int_{\mathbb{R}^d} u_{i,R}^2u_{j,R}^2 = \lim_{R \to +\infty} \int_{\mathbb{R}^d} u_{0,R}^2u_{j-i,R}^2 \leq \lim_{R \to +\infty} \left( \int_{\mathbb{R}^d} u_{0,R}^2u_{j-i,R}(dx) \right)^{2/3} \left( \int_{\mathbb{R}^d} u_{i,R}(dx) \right)^{1/3} = 0.
\]

(3.17)

Noting that \( \omega_i := \beta_i^{-1/2}U_{1,0} \) satisfies the equation \(-\Delta u = \beta_i u^3 \) in \( \mathbb{R}^d \), we have that
\[
  \int_{\mathbb{R}^d} |\nabla u_{j,R}|^2 = \int_{\mathbb{R}^d} \beta_j u_{j,R}^4 dx := C_j^* > 0,
\]
where \( C_j^* \) is a constant independent of \( R \). Consider the following algebraic \( N \)-equation about \( t_{j,R} \).
\[
  \int_{\mathbb{R}^d} \beta_j u_{j,R}^4 dx = t_{j,R} \beta_j \int_{\mathbb{R}^d} u_{j,R}^4 + \sum_{k \neq j} t_{k,R} \beta_k \int_{\mathbb{R}^d} u_{k,R}^2 u_{j,R}^2, \quad j = 1, \ldots, N. \tag{3.19}
\]

Let the matrix \( P_R := \left( \beta_{ij} \int_{\mathbb{R}^d} u_{i,R}^2 u_{j,R}^2 dx \right) \). In view of (3.17) and (3.18), for \( R \) large enough, we may assume that \( \sum_{k \neq j} |\beta_{kj}| \int_{\mathbb{R}^d} u_{k,R}^2 u_{j,R}^2 dx \leq \frac{1}{2} C_j^* \). Hence, by (3.18),
\[
  \beta_j \int_{\mathbb{R}^d} u_{j,R}^4 dx - \sum_{k \neq j} |\beta_{kj}| \int_{\mathbb{R}^d} u_{k,R}^2 u_{j,R}^2 dx \geq \frac{1}{2} C_j^*.
\]

By lemma 2.3, we deduce that \( \det(P_R) \geq \left( \frac{1}{2} \min_j C_j^* \right)^N > 0 \) which is independent of \( R \). On the other hand, since \( P_R^{-1} = \frac{\partial^2 I}{\partial \xi \partial \eta} \) and every element of \( P_R^{-1} \) is a continuous function of the variables \( \beta_{kj} \int_{\mathbb{R}^d} u_{i,R}^2 u_{j,R}^2 dx \) for \( 1 \leq k, j \leq N \), it follows from (3.17) that \( P_R^{-1} \to P_\infty^{-1} \) as \( R \to \infty \), where
\[
  P_R^{-1} := \begin{pmatrix}
    \frac{1}{C_j^*} & 0 & \cdots & 0 \\
    0 & \frac{1}{C_j^*} & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & \frac{1}{C_j^*}
  \end{pmatrix}.
\]

By (3.19), we have \( (t_{1,R}, \ldots, t_{N,R})^T = P_R^{-1} C := P_R^{-1} (C_1, \ldots, C_N)^T \). Let \( R \to \infty \); we get that \( (t_{j,R}, \ldots, t_{N,R})^T \to P_\infty^{-1} (C_1, \ldots, C_N)^T = (1, \ldots, 1)^T \). That is, \( t_{j,R} = 1 + o(1) \) as \( R \) is large enough, for all \( j = 1, \ldots, N \). By (3.19), we know that \( (t_{1,R}u_{1,R}, \ldots, t_{N,R}u_{N,R}) \in \mathcal{N} \) for all sufficiently large \( R \). It follows that
\[
  \Lambda \leq I(t_{1,R}u_{1,R}, \ldots, t_{N,R}u_{N,R}) = \frac{1}{4} \sum_{j=1}^N t_{j,R} \int_{\mathbb{R}^d} |\nabla u_{j,R}|^2 = \frac{1}{4} \sum_{j=1}^N t_{j,R} \beta_j^{-1} g^2.
\]
Letting $R \to \infty$, we get that
\[
A \leq \frac{1}{4} \sum_{j=1}^{N} \beta_{jj}^{-1} S^2.
\] 
(3.20)

On the other hand, for any $(u_1, ..., u_N) \in \mathcal{N}$, we see from $\beta_{jj} < 0, \forall i \neq j, 1 \leq i, j \leq N$, (1.11) and (1.24) that $\int_{\mathbb{R}^4} |\nabla u_j|^2 \leq \int_{\mathbb{R}^4} \beta_{jj} u_j^4 \text{dx} \leq \beta_{jj} S^{-2} (\int_{\mathbb{R}^4} |\nabla u_j|^2)^2$; hence, $\int_{\mathbb{R}^4} |\nabla u_j|^2 \geq \beta_{jj}^{-1} S^2$.

Combing these with (1.16), we get
\[
A \geq \frac{1}{4} \sum_{j=1}^{N} \beta_{jj}^{-1} S^2.
\] 
(3.21)

By (3.20) and (3.21), we see that $A = \frac{1}{4} \sum_{j=1}^{N} \beta_{jj}^{-1} S^2$. If we now assume that $A$ is attained by some $(u_1, ..., u_N) \in \mathcal{N}$, then $(|u_1|, ..., |u_N|) \in \mathcal{N}$ and $I(|u_1|, ..., |u_N|) = A$. By lemma 2.5, we get that $(|u_1|, ..., |u_N|)$ is a nontrivial solution of (1.10). By the maximum principle, we may assume that $u_j > 0$, and it follows that $\int_{\mathbb{R}^4} u_j^2 u_j^2 > 0$, and that $\int_{\mathbb{R}^4} \beta_{jj} u_j^2 u_j^2 < 0, i \neq j$. Inserting the equalities in (1.24), we have
\[
\int_{\mathbb{R}^4} |\nabla u_j|^2 < \int_{\mathbb{R}^4} \beta_{jj} u_j^4 \text{dx} \leq \beta_{jj} S^{-2} \left( \int_{\mathbb{R}^4} |\nabla u_j|^2 \right)^2;
\] 
(3.22)

therefore, $\int_{\mathbb{R}^4} |\nabla u_j|^2 > \beta_{jj}^{-1} S^2$. Moreover, we observe that
\[
A = I(u_1, ..., u_N) = \frac{1}{4} \sum_{j=1}^{N} \int_{\mathbb{R}^4} |\nabla u_j|^2 > \frac{1}{4} \sum_{j=1}^{N} \beta_{jj}^{-1} S^2,
\]
which is a contradiction. This completes the proof of (1) in theorem 1.3.

Next we prove part (2) of theorem 1.3. By the assumption of the theorem, (1.8) has a solution $(c_1, ..., c_N)$ satisfying $c_j > 0$ for all $1 \leq j \leq N$. Recall (3.3), we see that $(\sqrt{c_{11}} U_{c_1}, \sqrt{c_{22}} U_{c_2}, ..., \sqrt{c_{NN}} U_{c_N})$ is a nontrivial solution of (1.10) and
\[
A \leq I(\sqrt{c_{11}} U_{c_1}, ..., \sqrt{c_{NN}} U_{c_N}) = \sum_{j=1}^{N} c_j S^2.
\] 
(3.23)

Define $d_{ij} = (\int_{\mathbb{R}^4} u_i^4 j \text{dx})^{1/2}, i = 1, ..., N$, where $\{(u_{1,m}, ..., u_{N,m})\} \subset \mathcal{N}$ is a minimizing sequence of $A$. Then by (1.11), we have
\[
S d_{ij} \leq \int_{\mathbb{R}^4} |\nabla u_{ij}|^2 = \int_{\mathbb{R}^4} \beta_{jj} u_j^4 + \sum_{k \neq j} \int_{\mathbb{R}^4} \beta_{jk} u_k^2 u_j^2
\leq \beta_{jj} d_{ij}^2 + \sum_{k \neq j} \beta_{jk} d_{ij} d_{km}.
\] 
(3.24)

On the other hand, $\sum_{i=1}^{N} d_{ij} \leq 4 I(u_{1,m}, ..., u_{N,m}) \leq 4 \sum_{j=1}^{N} c_j S + o(1)$. Thus, we have
\[
\sum_{i=1}^{N} d_{ij} \leq \sum_{i=1}^{N} c_j S + o(1); \quad S \leq \beta_{ii} d_{ij} + \sum_{k \neq i} \beta_{jk} d_{km}.
\] 
(3.25)
Recalling (1.8), we see that (3.25) is equivalent to
\[
\begin{align*}
\sum_{j=1}^{N} (d_{j,m} - c_j S) & \leq a(1), \\
0 & \leq \beta_i (d_{j,m} - c_j S) + \sum_{k \neq i} \beta_{ik} (d_{k,m} - c_k S),
\end{align*}
\]
\[i = 1, \ldots, N.
\]
By lemma 2.4, we have \(d_{j,m} \to c_j S\) as \(m \to \infty\), and
\[
4A = \lim_{m \to \infty} 4I(u_{1,m}, \ldots, u_{N,m}) \geq \lim_{m \to \infty} S \sum_{i=1}^{N} d_{i,m} = 4 \sum_{i=1}^{N} c_i S.
\]
Combining this with (3.23), one has that
\[
A = \sum_{j=1}^{N} c_j S = I(\sqrt{\gamma_1} u_{1,y}, \ldots, \sqrt{\gamma_N} u_{N,y}).
\]
So, \((\sqrt{\gamma_1} u_{1,y}, \ldots, \sqrt{\gamma_N} u_{N,y})\) is a positive least energy solution of (1.10). This completes the proof of theorem 1.3.

3.4. Proof of theorem 1.4 (nonexistence) when some interactions are attractive while some others are repulsive, where \(\Omega = \mathbb{R}^4\) and all \(\lambda_j = 0\)

In this subsection, we shall prove theorem 1.4. For this purpose, we have to establish the following lemma. Let
\[
\mathcal{N} : = \left\{ (u_1, \ldots, u_N) \in D : u_j \neq 0, \int_{\mathbb{R}^4} |\nabla u_j|^2 \leq \int_{\mathbb{R}^4} \beta_j u_j^4 + \sum_{k \neq j} \beta_{jk} u_k^2 u_j^2 ; j = 1, \ldots, N \right\}
\]
and
\[
A' : = \inf_{(u_1, \ldots, u_N) \in \mathcal{N}'} \frac{1}{4} \sum_{j=1}^{N} \int_{\mathbb{R}^4} |\nabla u_j|^2 dx.
\]
Then we have

**Lemma 3.1.** If \(\beta_{ij} \geq 0, \beta_{kk} > 0, \forall k, i \neq j, 1 \leq i, j, k \leq N\), the matrix \(B\) is invertible, and the sum of each column of \(B^{-1}\) is greater than 0, then \(A'\) is attained by
\[
(\sqrt{\gamma_1} U_{1,y}, \sqrt{\gamma_2} U_{2,y}, \ldots, \sqrt{\gamma_N} U_{N,y}),
\]
where \(\sum_{k=1}^{N} \beta_{jk} c_k = 1, j = 1, \ldots, N\).

**Proof.** We can follow the proof of (3) in theorem 1.3 by replacing \(\mathcal{N}\) and \(A\) with \(\mathcal{N}'\) and \(A'\) respectively. We need only replace `\(\leq\)` in (3.24) by the inequality `\(\leq\)` and other details remain unchanged.

Combing (3) in theorem 1.3 with lemma 3.1, we have the next useful property.

**Corollary 3.1.** If \(\beta_{ij} \geq 0, \beta_{kk} > 0, \forall k, i \neq j, 1 \leq i, j, k \leq N\) and the matrix \(B\) is invertible and the sum of each column of \(B^{-1}\) is greater than 0, then \(A' = A\), that is,
\[
\inf_{(u_1, \ldots, u_N) \in \mathcal{N}'} \frac{1}{4} \sum_{j=1}^{N} \int_{\mathbb{R}^4} |\nabla u_j|^2 dx = \inf_{(u_1, \ldots, u_N) \in \mathcal{N}} \frac{1}{4} \sum_{j=1}^{N} \int_{\mathbb{R}^4} |\nabla u_j|^2 dx.
\]
Now we are ready to prove theorem 1.4.

**Proof of theorem 1.4.** Assume that \((u_1, \ldots, u_N) \in \mathcal{N}\). Without loss of generality, we may assume that \(i_0 = 1\); then the energy \(I(u_1, \ldots, u_N)\) can be divided into two parts:

\[
I(u_1, \ldots, u_N) = \frac{1}{2} \int_{\mathbb{R}^4} |\nabla u_1|^2 \, dx - \frac{1}{4} \beta_{11} \int_{\mathbb{R}^4} u_1^4 - \frac{1}{2} \sum_{j=2}^{N} \beta_{ij} \int_{\mathbb{R}^4} u_i^2 u_j^2 + I_1(u_2, \ldots, u_N),
\]

where

\[
I_1(u_2, \ldots, u_N) = \sum_{j=2}^{N} \left( \frac{1}{2} \int_{\mathbb{R}^4} |\nabla u_j|^2 \, dx - \frac{1}{4} \beta_{jj} \int_{\mathbb{R}^4} u_j^4 \right) - \frac{1}{4} \sum_{2 \leq i \neq j \leq N} \beta_{ij} \int_{\mathbb{R}^4} u_i^2 u_j^2.
\]

Since \(\beta_{ij} < 0\) for \(j > 1\) and \((u_1, \ldots, u_N) \in \mathcal{N}\), we have that

\[
\int_{\mathbb{R}^4} |\nabla u_1|^2 - \beta_{11} \int_{\mathbb{R}^4} u_1^4 = \sum_{j=2}^{N} \beta_{ij} \int_{\mathbb{R}^4} u_i^2 u_j^2 \leq 0,
\]

that is, \(\int_{\mathbb{R}^4} |\nabla u_1|^2 \leq \beta_{11} \int_{\mathbb{R}^4} u_1^4\). Combining this with the Sobolev inequality, we get that

\[
S(\int_{\mathbb{R}^4} u_1^4)^{1/2} \leq \int_{\mathbb{R}^4} |\nabla u_1|^2 \leq \beta_{11} \int_{\mathbb{R}^4} u_1^4;
\]

hence, \(\int_{\mathbb{R}^4} u_1^4 \geq \beta_{11}^{-1} S^2\) and \(\int_{\mathbb{R}^4} |\nabla u_1|^2 \geq \beta_{11}^{-1} S^2\). Therefore,

\[
\frac{1}{4} \beta_{11} \int_{\mathbb{R}^4} u_1^4 \leq \frac{1}{4} \beta_{11} S^{-2} \left( \int_{\mathbb{R}^4} |\nabla u_1|^2 \right)^2 \leq \frac{1}{4} \beta_{11}^{-1} S^2.
\]

By these inequalities with (3.29) and \(\beta_{ij} < 0\) for \(j > 1\), we have

\[
I(u_1, \ldots, u_N) \geq \frac{1}{2} \int_{\mathbb{R}^4} |\nabla u_1|^2 \, dx - \frac{1}{4} \beta_{11} \int_{\mathbb{R}^4} u_1^4 \, dx + I_1(u_2, \ldots, u_N),
\]

\[
\geq \frac{1}{4} \beta_{11}^{-1} S^2 + I_1(u_2, \ldots, u_N).
\]

By applying theorem 1.3, we may let \((u_0^1, \ldots, u_0^N)\) be the least energy solution of \(I_1\), i.e.

\[
I_1(u_0^1, \ldots, u_0^N) = \inf_{(v_2, \ldots, v_N) \in \mathcal{N}_{\mathbb{R}}^N} I_1(v_2, \ldots, v_N) := C_1,
\]

where
\[ N_0 := \left\{ (u_2, \ldots, u_N) \in D^{1,2}(\mathbb{R}^4) \times \cdots \times D^{1,2}(\mathbb{R}^4) : u_j \neq 0, \int_{\mathbb{R}^4} |\nabla u_j|^2 \right\}. \]  
(3.33)

Let \( u_j^* := \beta_{11}^{1/2} U_{1,0}(x - R\epsilon_1) \) and \( u_j^* = u_j^0 \) for \( j = 2, \ldots, N \). Then, by theorem 1.3-(3), we have \( u_j^* = k_j U_{j,0} \) for some positive constants \( k_j, j = 2, \ldots, N \). By the same proof of (3.17), we have

\[ \int_{\mathbb{R}^4} (u_1^*)^2 = 0 \quad \text{as} \quad R \to \infty, \quad \text{for all} \quad j = 2, \ldots, N. \]  
(3.34)

Now consider the following algebraic \( N \)-equation about \( t_j, R \),

\[ \int_{\mathbb{R}^4} |\nabla u_j|^2 = t_j R \int_{\mathbb{R}^4} \beta_j (u_j^*)^4 + \sum_{k \neq j} t_{k,R} \beta_{jk} \int_{\mathbb{R}^4} (u_k^*)^2 (u_j^*)^2, \quad j = 1, \ldots, N. \]  
(3.35)

Let the \( N \times N \)-matrix \( P_{R,N} := \left( \beta_{jk} \int_{\mathbb{R}^4} (u_k^*)^2 (u_j^*)^2 \right) \). Recall our assumption that \( B^1 \) is positively definite; by the proof of (2.3), we know that the \((N - 1) \times (N - 1)\)-matrix \( \tilde{P}_{N-1} \) is positively definite, and therefore \( \det(\tilde{P}_{N-1}) > 0 \) independently of \( R \), where \( \tilde{P}_{N-1} \) denotes the matrix constructed by removing the first column and the first row of \( P_{R,N} \). Set

\[ \tilde{P} := \left( \begin{array}{cc} \int_{\mathbb{R}^4} \beta_{11} (u_1^*)^4 & 0 \\ 0 & \tilde{P}_{N-1} \end{array} \right). \]  
(3.36)

Then \( \det(\tilde{P}) = \int_{\mathbb{R}^4} \beta_{11} (u_1^*)^4 dx \cdot \det(\tilde{P}_{N-1}) > 0 \) independently of \( R \). Recall (3.34); for large enough \( R \), we can assume that \( \det(P_{R,N}) \geq \frac{1}{2} \det(\tilde{P}) > 0 \); therefore, the linear system (3.35) has a unique solution for large enough \( R \). Next, we estimate the asymptotic behavior of \( t_j, R \) as \( R \to \infty \). Note that \( (P_{R,N})^{-1} = \frac{P_{R,N}}{\det(P_{R,N})} \), and every element of \( P_{R,N}^{-1} \) is a continuous function about the variables \( \beta_{jk} \int_{\mathbb{R}^4} (u_k^*)^2 (u_j^*)^2 dx \) for \( 1 \leq k, j \leq N \). Combining this with (3.34), we get that \( P_{R,N}^{-1} \to P^{-1} \) as \( R \to \infty \). By (3.35), we obtain

\[ (t_{1,R}, t_{2,R}, \ldots, t_{N,R})^T = P_{R,N}^{-1} \left( \int_{\mathbb{R}^4} |\nabla u_1|^2, \int_{\mathbb{R}^4} |\nabla u_2|^2, \ldots, \int_{\mathbb{R}^4} |\nabla u_N|^2 \right)^T \]
\[ \to P^{-1} \left( \int_{\mathbb{R}^4} |\nabla u_1|^2, \int_{\mathbb{R}^4} |\nabla u_2|^2, \ldots, \int_{\mathbb{R}^4} |\nabla u_N|^2 \right)^T \quad \text{as} \quad R \to \infty. \]  
(3.37)

Notice that \( u_j^* \in N_0 \) for \( j = 2, \ldots, N \), and that \( \int_{\mathbb{R}^4} |\nabla u_j|^2 = \int_{\mathbb{R}^4} \beta_{11} (u_j^*)^4 \); together with (3.36), we obtain that

\[ P_{R,N}^{-1} \left( \int_{\mathbb{R}^4} |\nabla u_1|^2, \int_{\mathbb{R}^4} |\nabla u_2|^2, \ldots, \int_{\mathbb{R}^4} |\nabla u_N|^2 \right)^T = (1, 1, \ldots, 1)^T. \]  
(3.38)

From (3.37) and (3.38), we have \( t_j, R \to 1 \) as \( R \to \infty \) for \( j = 1, \ldots, N \). On the other hand, by (3.35), we must have \( (\sqrt{t_{1,R}} u_1^*, \ldots, \sqrt{t_{N,R}} u_N^*) \in N \). Hence, combining this with (3.34) and \( \int_{\mathbb{R}^4} |\nabla u_j|^2 = \int_{\mathbb{R}^4} \beta_{11} (u_j^*)^4 \), we have
\[ C := \inf_{(u_1, \ldots, u_N) \in \mathcal{N}} I(u_1, \ldots, u_N) \]
\[ \leq I(\sqrt{t_1 R}u_1^1, \ldots, \sqrt{t_N R}u_N^N) \]
\[ = \frac{1}{2} \int_{\mathbb{R}^4} t_1 R|\nabla u_1^1|^2 \, dx - \frac{1}{4} \beta_{11} \int_{\mathbb{R}^4} t_1 R(u_1^1)^4 \]
\[ - \left( \frac{1}{2} \sum_{j=2}^N \beta_{1j} \int_{\mathbb{R}^4} t_1 R(u_1^j)^2 (u_j^2)^2 + I_1(u_2^2, \ldots, u_N^N) \right) \]
\[ \leq \frac{1}{4} \beta_{11}^{-1} S^2 + C_1, \text{ as } R \to \infty \text{ hence } t_{1j} \to 1, j = 1, 2, \ldots, N, \] (3.39)

where \( C_1 \) is defined in (3.32). Combine with (3.31), (3.32) and (3.39), we obtain \( C = \frac{1}{4} \beta_{11}^{-1} S^2 + C_1 \). Finally, we need to show that \( C \) is not attained. Firstly, from (3.30) and (3.11), we have

\[ \frac{1}{2} \int_{\mathbb{R}^4} |\nabla u_1|^2 \, dx - \frac{1}{4} \beta_{11} \int_{\mathbb{R}^4} u_1^4 \geq \frac{1}{4} \beta_{11}^{-1} S^2. \] (3.40)

Assume to the contrary that \( C \) is attained by some \((u_1^0, \ldots, u_N^0) \in \mathcal{N} \) satisfying \( u_j^0 > 0 \) for \( j = 1, \ldots, N \). Then we see from \( \beta_{1j} < 0 \) for \( j = 2, \ldots, N \) that \((u_2^0, \ldots, u_N^0) \in \mathcal{N}'_0\), where

\[ \mathcal{N}'_0 := \left\{ (u_2, \ldots, u_N) \in D^1 \times \cdots \times D^1 \right\} : u_j \neq 0, \]
\[ \int_{\mathbb{R}^4} |\nabla u_j|^2 \leq \int_{\mathbb{R}^4} \beta_{1j} u_j^4 + \sum_{k \neq j}^N \int_{\mathbb{R}^4} \beta_{kj} u_k^2 u_j^2 ; \quad j = 2, \ldots, N. \] (3.41)

Recall that \( \beta_{ij} > 0 \) for \( 2 \leq i \neq j \leq N \), by corollary 3.1,

\[ I_1(u_2^0, \ldots, u_N^0) \geq \inf_{(u_2, \ldots, u_N) \in \mathcal{N}'_0} I_1(u_2, \ldots, u_N) = \inf_{(u_2, \ldots, u_N) \in \mathcal{N}'_0} I_1(u_2, \ldots, u_N). \]

Then by (3.29), (3.40) and (3.32),

\[ C = I(u_1^0, \ldots, u_N^0) \geq \frac{1}{4} \beta_{11}^{-1} S^2 + I_1(u_2^0, \ldots, u_N^0) \geq \frac{1}{4} \beta_{11}^{-1} S^2 + C_1, \] (3.42)
a contradiction! \( \square \)

4. **The subcritical case: \( n = 2, 3 \)**

4.1. **Proof of theorem 1.6 (existence), where \( \Omega = \mathbb{R}^n, n = 2, 3 \)**

**Proof.** Recalling that \( \mathcal{B} \) is invertible, and that the sum of each line of \( \mathcal{B}^{-1} \) is greater than 0, it follows that equation (1.8) has a solution \((c_1, \ldots, c_N)\) satisfying \( c_j > 0 \) for \( 1 \leq j \leq N \). As a consequence, \((\sqrt{c_1} \omega_{\lambda, 1}, \sqrt{c_2} \omega_{\lambda, 1}, \ldots, \sqrt{c_N} \omega_{\lambda, 1})\) is a nontrivial solution of (1.18), and \( I(\sqrt{c_1} \omega_{\lambda, 1}, \ldots, \sqrt{c_N} \omega_{\lambda, 1}) = \sum_{j=1}^N \omega_{\lambda, j} T_{\lambda, 1} \), where \( T_{\lambda, 1} \) is defined in (1.20). Let \( \{(v_{1m}, \ldots, v_{Nm})\} \subset \mathcal{N} \) be a minimizing sequence of \( T \) (defined in (1.25))—that is, \( I(v_{1m}, \ldots, v_{Nm}) \to T \) as \( m \to \infty \). Define \( z_{im} = (\int_{\mathbb{R}^n} v_{im}^4 \, dx)^{1/2}, \quad i = 1, \ldots, N \). Then by
(1.22), we have
\[ 2\sqrt{T_{1,1}z_{1,m}} \leq \int_{\mathbb{R}^n} (|\nabla v_{1,m}|^2 + \lambda v_{1,m}^2) = \int_{\mathbb{R}^n} \beta_{1,1}v_{1,m}^2 + \sum_{k \neq 1} \int_{\mathbb{R}^n} \beta_{k,1}v_{k,1}^2 v_{1,m} \]
\[ \leq \beta_{1,1}^2 z_{1,m}^2 + \sum_{k \neq 1} \beta_{k,1}z_{1,m} z_{k,1}. \]

(4.1)

On the other hand, \( 2\sqrt{T_{1,1}} \sum_{i=1}^N z_{i,m} \leq 4H(v_{1,m}, ..., v_{N,m}) \leq 4 \sum_{i=1}^N c_i T_{i,1} + o(1) \). Recalling (1.8), we have
\[ \begin{cases} \sum_{i=1}^N (z_{i,m} - c_i 2\sqrt{T_{i,1}}) \leq o(1), \\ 0 \leq \beta_i (z_{i,m} - c_i 2\sqrt{T_{i,1}}) + \sum_{k \neq i} \beta_{ki}(z_{k,m} - c_i 2\sqrt{T_{i,1}}), \\ i = 1, \cdots, N. \end{cases} \]

(4.2)

By lemma 2.4, we deduce that \( z_{i,m} \to c_i 2\sqrt{T_{i,1}} \) as \( m \to \infty \) and that
\[ 4T = \lim_{m \to \infty} 4H(v_{1,m}, ..., v_{N,m}) \geq \lim_{m \to \infty} 2\sqrt{T_{1,1}} \sum_{i=1}^N z_{i,m} = 4 \sum_{i=1}^N c_i T_{i,1}. \]

(4.3)

It follows that \( T = \sum_{i=1}^N c_i T_{i,1} = I(\sqrt{\epsilon_1 \omega_{1,1}}, ..., \sqrt{\epsilon_N \omega_{N,1}}) \). The proof is thus complete.

4.2. Proof of theorem 1.7 (uniqueness), where \( \Omega = \mathbb{R}^n, n = 2, 3 \)

Proof. Let \((v_{1,0}, ..., v_{N,0})\) be a least energy solution of (1.18); then \( v_{j,0} > 0 \) for \( j = 1, ..., N \), by the strong maximum principle. Throughout this subsection, for each fixed pair \((m, l)\) with \( 1 \leq m, l \leq N \), we view \( \beta_{ml} \) as a variable. Recall the definitions of \( I, \mathcal{N} \) and \( T \); these all depend on \( \beta_{ml} \), and we use notations \( I_{\beta_{ml}}, \mathcal{N}_{\beta_{ml}} \) and \( T(\beta_{ml}) \) in this proof. For each \( j \), we define the following function of \((t_1, ..., t_N)\):
\[ h_j(t_1, ..., t_N) = \int_{\mathbb{R}^n} t_j \partial \beta_j v_{j,0}^2 + \sum_{k \neq j} \int_{\mathbb{R}^n} t_k \partial \beta_{kj} v_{k,0}^2 v_{j,0} - \int_{\mathbb{R}^n} (|\nabla v_{j,0}|^2 + \lambda v_{j,0}^2). \]

(4.4)

Then we have \( h_j(1, ..., 1) = 0 \), and \( \frac{\partial h_j}{\partial \beta_{ml}} = \partial \beta_{ml} \int_{\mathbb{R}^n} v_{m,0}^2 v_{l,0}^2 \). Let the matrix \( H := \left( \frac{\partial h_i}{\partial \beta_{ml}} \right) = (H_{ij}) \). Since \( B \) is positively or negatively definite, so is \( H \), and \( \det(H) \neq 0 \). Therefore, by the implicit function theorem, the functions \( t_j(\beta_{ml}) \) are well-defined for \( \beta_{ml} \in (\beta_{ml} - \delta_1, \beta_{ml} + \delta_1) \), and are of class \( C^1 \) for some \( 0 < \delta_1 \leq \delta \). Moreover, since \( t_j(\beta_{ml}) = 1, j = 1, ..., N \), we may assume that \( t_j(\beta_{ml}) > 0 \) for all \( \beta_{ml} \in (\beta_{ml} - \delta_1, \beta_{ml} + \delta_1) \) by choosing a small value of \( \delta_1 \). Observing that \( h_1(t_1(\beta_{ml}), ..., t_N(\beta_{ml})), \beta_{ml} \equiv 0 \), we obtain \( \sum_{j=1}^N \frac{\partial h_j}{\partial \beta_{ml}} t_j(\beta_{ml}) = -\frac{\partial h}{\partial \beta_{ml}} \), i.e. \( t_j'(\beta_{ml}) = -\sum_{k=1}^N \frac{\partial h_k}{\partial \beta_{ml}} H_{kj} \), where \( (H^p) \) denotes the inverse of the matrix \( H \). On the other hand, by (4.4), we have
\[ \frac{\partial h_j}{\partial \beta_{ml}} (1, ..., 1) = \delta_{km} \int_{\mathbb{R}^n} v_{m,0}^2 v_{l,0}^2. \]
Inserting the obtained results into (4.2), one has
\[ t'_j(\beta_{ml}) = -\sum_k \delta_{km} \int_{\mathbb{R}^n} v_{m,0}^2 v_{l,0}^2 dx H^j = -\int_{\mathbb{R}^n} v_{m,0}^2 v_{l,0}^2 dx H^j. \]

By the Taylor expansion, we see that \( t'_j(\beta_{ml}) = 1 + t'_j(\beta_{ml})(\beta_{ml} - \beta_{ml}) + O((\beta_{ml} - \beta_{ml})^2) \).

Note that \( b_j(1, \beta_{ml}), \ldots, b_j(\beta_{ml}), \beta_{ml} \equiv 0 \), which implies that
\[ \sqrt{t_1(\beta_{ml})v_{1,0}}, \ldots, \sqrt{tv(\beta_{ml})v_{N,0}} \in N_{\beta ml}. \]
hence
\[ T(\beta_{ml}) \leq T_{\beta ml} \left( \sqrt{t_1(\beta_{ml})v_{1,0}}, \ldots, \sqrt{tv(\beta_{ml})v_{N,0}} \right) \]
\[ = \frac{1}{4} \sum_{j=1}^{N} t'_j(\beta_{ml}) \int_{\mathbb{R}^n} (|\nabla v_{j,0}|^2 + \lambda v_{j,0}^2) dx \]
\[ = T(\beta_{ml}) + \frac{1}{4} \Theta(\beta_{ml} - \beta_{ml}) + O((\beta_{ml} - \beta_{ml})^2), \quad (4.5) \]

where
\[ \Theta := \sum_{j=1}^{N} t'_j(\beta_{ml}) \int_{\mathbb{R}^n} (|\nabla v_{j,0}|^2 + \lambda v_{j,0}^2) \]
\[ = \sum_{j=1}^{N} t'_j(\beta_{ml}) \left( \int_{\mathbb{R}^n} \beta_j v_{j,0}^4 dx + \sum_{k \neq j} \beta_{kj} \int_{\mathbb{R}^n} v_{l,0}^2 v_{j,0}^2 dx \right) \]
\[ = -\int_{\mathbb{R}^n} v_{m,0}^2 v_{l,0}^2 \sum_{j=1}^{N} \sum_{k \neq j} H_{mj}(H_{kj} + \sum_{l \neq k} H_{jlk}) \]
\[ = -\int_{\mathbb{R}^n} v_{m,0}^2 v_{l,0}^2 \sum_{k=1}^{N} \delta_{km} = -\int_{\mathbb{R}^n} v_{m,0}^2 v_{l,0}^2. \quad (4.6) \]

It deduces from (4.5) that
\[ \frac{T(\beta_{ml}) - T(\beta_{ml})}{\beta_{ml} - \beta_{ml}} > \frac{\Theta}{4} + O(\beta_{ml} - \beta_{ml}) \quad (4.7) \]
as \( \beta_{ml} \gg \beta_{ml} \) and \( T'(\beta_{ml}) \geq 0 \). Similarly, we have \( \frac{T(\beta_{ml}) - T(\beta_{ml})}{\beta_{ml} - \beta_{ml}} \leq \frac{\Theta}{4} + O(\beta_{ml} - \beta_{ml}) \) as \( \beta_{ml} \gg \beta_{ml} \), that is, \( T'(\beta_{ml}) \leq \frac{\Theta}{4} \). Hence, \( T'(\beta_{ml}) = -\frac{\Theta}{4} = -\frac{1}{4} \int_{\mathbb{R}^n} v_{m,0}^2 v_{l,0}^2. \) On the other hand, by lemma 2.1 and theorem 1.6, we have \( T'(\beta_{ml}) = -c_m c_l T_{\lambda,1} = -\frac{1}{4} c_m c_l \int_{\mathbb{R}^n} \omega_{\lambda,1}. \) Hence, \( \int_{\mathbb{R}^n} v_{m,0}^2 v_{l,0}^2 = c_m c_l \int_{\mathbb{R}^n} \omega_{\lambda,1}. \) Let \( v_{i,0} := \frac{1}{\sqrt{3}} v_{i,0} \) for \( i = 1, \ldots, N \). Since \( \Sigma_{k} \beta_{kj} c_k = 1 \) and \( (v_{1,0}, \ldots, v_{N,0}) \in N \), we obtain that
\[ \int_{\mathbb{R}^n} (|\nabla v_{j,0}|^2 + \lambda v_{j,0}^2) \, dx = \frac{1}{C_j} \int_{\mathbb{R}^n} (|\nabla v_{j,0}|^2 + \lambda v_{j,0}^2) \, dx \]

\[ = \frac{1}{C_j} \left( \int_{\mathbb{R}^n} \beta_j v_{j,0}^4 \, dx + \sum_{k \neq j} \beta_k c_k \int_{\mathbb{R}^n} v_{k,0}^2 v_{j,0}^2 \, dx \right) \]

\[ = \frac{1}{C_j} (\beta_j c_j^2 + \sum_{k \neq j} \beta_k c_k c_j) \int_{\mathbb{R}^n} \omega_{\lambda,1}^2 \, dx \]

\[ = (\beta_j c_j + \sum_{k \neq j} \beta_k c_k) \int_{\mathbb{R}^n} \omega_{\lambda,1}^2 \, dx = \int_{\mathbb{R}^n} v_{j,0}^4 \, dx. \quad (4.8) \]

By (1.22), we have \( \frac{1}{4} \int_{\mathbb{R}^n} (|\nabla v_{j,0}|^2 + \lambda v_{j,0}^2) \, dx \geq T_{\lambda,1}, \ j = 1, \ldots, N \). Hence,

\[ T = \sum_{j=1}^{N} c_j T_{\lambda,1} = \frac{1}{4} \sum_{j=1}^{N} \int_{\mathbb{R}^n} (|\nabla v_{j,0}|^2 + \lambda v_{j,0}^2) \, dx \]

\[ = \frac{1}{4} \sum_{j=1}^{N} c_j \int_{\mathbb{R}^n} (|\nabla v_{j,0}|^2 + \lambda v_{j,0}^2) \, dx \geq \sum_{j=1}^{N} c_j T_{\lambda,1}. \]

which implies that \( \frac{1}{4} \int_{\mathbb{R}^n} (|\nabla v_{j,0}|^2 + \lambda v_{j,0}^2) \, dx = T_{\lambda,1}, \ j = 1, \ldots, N \). Furthermore, we know that each of \( v_{j,0} \) \((j = 1, 2, \ldots, N)\) is the positive least energy solutions of (1.21). Note that the uniqueness of the solution to the equation (1.21), we have \( v_{j,0} = \omega_{\lambda,1}, j = 1, 2, \ldots, N \). Hence \( (v_1,0, \ldots, v_N,0) = (\sqrt{n} \omega_{\lambda,1}, \ldots, \sqrt{n} \omega_{\lambda,1}) \), where \( \omega_{\lambda,1} \) is the unique positive (and radially symmetric) least energy solution of (1.21).

\[ \square \]

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