NOTE: Textbooks written for U.S.A. markets (such as our textbook) often use FPS (foot-pound-second) units. In the FPS (or "Imperial") system of units, the unit of length is the foot (ft), and 1 ft = 12 inches (in). [The SI or "metric" unit of length is the metre (m).] The unit of time is the second (s) [the same as the SI unit of time]. The FPS unit of force is the pound (lb) [the SI unit of force is the Newton (N)]. Weight is a type of force, so it is measured in pounds, weight = \( mg \) where \( g \) is acceleration due to gravity \( (g = 32 \text{ ft} \cdot \text{s}^{-2} \text{ in FPS units, approximately, at the surface of the earth}) \) and \( m \) is the mass, measured in FPS units of slugs or \( \text{lb} \cdot \text{s}^2 \cdot \text{ft}^{-1} \) [the SI unit of mass is the kilogram (kg)]. In FPS units, "an object weighs 5 lb" means \( mg = 5 \text{ lb} \), where \( g = 32 \text{ ft} \cdot \text{s}^{-2} \), so \( m = \frac{5}{32} \) slugs; a mass of \( \frac{5}{32} \) slugs weighs 5 lb. [But in SI units, a mass of 5 kg weighs \( mg = (5 \text{ kg})(9.8 \text{ m} \cdot \text{s}^{-2}) = 49 \text{ N} \) (approximately, at the surface of the earth).]

1. A mass that weighs 3 lb stretches a spring 3 in. The mass is pushed upwards from its equilibrium position, contracting the spring a distance of 1 in, then set in motion with a downwards velocity of 2 ft/s. If the motion is undamped, find the natural frequency \( \omega_0 \), amplitude \( R \) and phase \( \delta \) of the downwards displacement \( u(t) = R \cos(\omega_0 t - \delta) \).

2. A mass weighing 16 lb stretches a spring 3 in. The mass is attached to a viscous damper with damping constant of 2 lb \cdot \text{s}/\text{ft}. If the mass is set in motion from its equilibrium position with a downwards velocity of 3 in/s, find its downwards displacement at any time \( t \geq 0 \). Write the displacement in the form \( u(t) = R e^{\lambda t} \cos(\mu t - \delta) \) and determine \( R, \lambda, \delta \) and the quasi-frequency \( \mu \), using exact expressions.

3. A mass of 5 kg stretches a spring 10 cm. The mass is acted on by an external force of \( 10 \sin(t/2) \text{ N} \) and moves in a medium that imparts a viscous force of 2 N when the speed of the mass is 4 cm/s. If the mass is set in motion from its equilibrium position with an initial downwards velocity of 3 cm/s:
   (a) Find the position of the mass at any time \( t \geq 0 \).
   (b) Identify the transient and steady-state parts of the solution.
   (c) If the given external force is replaced by a force of \( 2 \cos(\omega t) \) with forcing frequency \( \omega \), find the value of \( \omega \) for which the amplitude of the forced response (or "steady-state") is the maximum possible.

4. An undamped mass-spring system with a mass that weighs 6 lb and a spring constant of 12 lb/ft is suddenly set in motion at \( t = 0 \) by an external force of \( 4 \cos(7t) \text{ lb} \). Determine the displacement of the mass at any time \( t \geq 0 \), and write the displacement as a product of two trigonometric functions of different frequencies. Sketch the graph of the position versus \( t \), showing the amplitude modulation with the slow frequency.
1. A mass that weighs 3 lb stretches a spring 3 in. The mass is pushed upwards from its equilibrium position, contracting the spring a distance of 1 in, then set in motion with a downwards velocity of 2 ft/s. If the motion is undamped, find the natural frequency $\omega_0$, amplitude $R$ and phase $\delta$ of the downwards displacement $u(t) = R\cos(\omega_0 t - \delta)$.  

Solution: In FPS units, since $L = 3$ inches $= \frac{1}{4}$ foot, and the weight is $mg = 3$ lb, the spring constant is $k = mg/L = (3\text{ lb})/(\frac{1}{4}\text{ ft}) = 12$ lb/ft. The mass is $m = (mg)/g = (3\text{ lb}/(32\text{ ft}/s^2)) = \frac{3}{32}$ slug, and there is no damping: $\gamma = 0$. If $u(t)$ is downwards displacement, in ft, from the equilibrium position, the differential equation to solve is

$$\frac{3}{32} u'' + 12 u = 0,$$

with initial conditions $u(0) = -\frac{1}{12}, \quad u'(0) = 2$. The linear second-order differential equation with constant coefficients is homogeneous, its characteristic equation is $\frac{3}{32} r^2 + 12 = 0$, which has purely imaginary roots $r_{1,2} = \pm i\sqrt{(32)(12)/3} = \pm i8\sqrt{2} \approx \pm i 11.3137$, and the general solution of the differential equation is

$$u(t) = c_1 \cos(8\sqrt{2} t) + c_2 \sin(8\sqrt{2} t),$$

where $c_1$ and $c_2$ are arbitrary constants. Using the initial conditions

$$u(0) = -\frac{1}{12}, \quad u'(0) = 2$$

gives

$$c_1 = -\frac{1}{12} \quad (\approx 0.0833), \quad c_2 = \frac{1}{4\sqrt{2}} = \frac{\sqrt{2}}{8} \quad (\approx 0.1768),$$

thus the position of the mass at any time $t$ is given by

$$u(t) = -\frac{1}{12} \cos(8\sqrt{2} t) + \frac{1}{4\sqrt{2}} \sin(8\sqrt{2} t).$$

This can be written in the form

$$u(t) = R\cos(\omega_0 t - \delta),$$

where the natural frequency is

$$\omega_0 = 8\sqrt{2} \quad (\approx 11.3137) \text{ s}^{-1},$$

the amplitude is

$$R = \sqrt{\left(-\frac{1}{12}\right)^2 + \left(\frac{1}{4\sqrt{2}}\right)^2} = \frac{\sqrt{11}}{12} \quad (\approx 0.2764) \text{ ft},$$

and the phase $\delta$ satisfies

$$\tan \delta = \frac{\frac{1}{4\sqrt{2}}}{-\frac{1}{12}} = -\frac{3}{\sqrt{2}}$$
with $\delta$ in Quadrant II. Therefore

$$\delta = \pi + \arctan \left( -\frac{3}{\sqrt{2}} \right) \approx 2.0113.$$  

2. A mass weighing 16 lb stretches a spring 3 in. The mass is attached to a viscous damper with damping constant of 2 lb $\cdot$ s/ft. If the mass is set in motion from its equilibrium position with a downwards velocity of 3 in/s, find its downwards displacement at any time $t \geq 0$. Write the displacement in the form $u(t) = Re^{\lambda t} \cos(\mu t - \delta)$ and determine $R$, $\lambda$, $\delta$ and the quasi-frequency $\mu$, using exact expressions.

**Solution:** In FPS units, weight is $mg = 16$ lb, the mass is $m = (mg)/g = (16 \text{ lb})/(32 \text{ ft/s}^2) = 1/2$ slug, the spring constant is $k = mg/L = (16 \text{ lb})/(1/4 \text{ ft}) = 64$ lb/ft, and the damping constant is given as $\gamma = 2$ lb$\cdot$s/ft. If $u(t)$ is downwards displacement, in ft, from the equilibrium position, the differential equation to solve is

$$\frac{1}{2} u'' + 2u' + 64u = 0,$$

with initial conditions $u(0) = 0$, $u'(0) = \frac{1}{4}$. This is a linear second-order differential equation with constant coefficients, and it is homogeneous. The characteristic equation is $\frac{1}{2}r^2 + 2r + 64 = 0$, which has the roots $r_{1,2} = -2 \pm 2\sqrt{31}$ and the general solution of the differential equation is

$$u(t) = c_1 e^{-2t} \cos(2\sqrt{31}t) + c_2 e^{-2t} \sin(2\sqrt{31}t),$$

where $c_1$, $c_2$ are arbitrary constants. Using the initial conditions

$$u(0) = 0, \quad u'(0) = \frac{1}{4}$$

gives

$$c_1 = 0, \quad c_2 = \frac{1}{8\sqrt{31}},$$

thus the position of the mass at any time $t$ (in s) is given by

$$u(t) = \frac{1}{8\sqrt{31}} e^{-2t} \sin(2\sqrt{31}t).$$

This can be written in the form $u(t) = Re^{\lambda t} \cos(\mu t - \delta)$ where the amplitude is

$$R = \frac{1}{8\sqrt{31}} \approx 0.02245 \text{ ft},$$

and the exponential growth rate is

$$\lambda = -2 \text{ s}^{-1}$$

(a negative exponential growth rate means exponential decay). The phase $\delta$ satisfies $R \cos \delta = 0$ and $R \sin \delta = \frac{1}{8\sqrt{31}} > 0$, therefore

$$\delta = \frac{\pi}{2}$$

(alternatively, $\sin \theta = \cos \left( \theta - \frac{\pi}{2} \right)$ by a trigonometric identity), and the quasi-frequency is

$$\mu = 2\sqrt{31} \approx 11.1355 \text{ s}^{-1}.$$
3. A mass of 5 kg stretches a spring 10 cm. The mass is acted on by an external force of 
$10 \sin(t/2)$ N and moves in a medium that imparts a viscous force of 2 N when the speed of 
the mass is 4 cm/s. If the mass is set in motion from its equilibrium position with an initial 
downwards velocity of 3 cm/s:
(a) Find the position of the mass at any time $t \geq 0$.
(b) Identify the transient and steady-state parts of the solution.
(c) If the given external force is replaced by a force of $2 \cos(\omega t)$ with forcing frequency $\omega$, 
find the value of $\omega$ for which the amplitude of the forced response (or "steady-state") is the 
maximum possible.

Solution: In SI units, the mass is $m = 5$ kg, the spring constant is $k = mg/L = (5)(9.8)/(0.1) = 
490$ N/m, and the damping constant is $\gamma = 2/0.04 = 50$ N·s/m (note that 4 cm/s = 0.04 
m/s). The differential equation for the downwards displacement $u(t)$ from equilibrium (in 
m) is

$$5u'' + 50u' + 490u = 10 \sin(t/2), \quad u(0) = 0, \quad u'(0) = 0.93.$$ 

with initial conditions $u(0) = 0$, $u'(0) = 0.03 = \frac{3}{100}$. The differential equation is linear, 
second-order with constant coefficients, and it is nonhomogeneous.

(a) The corresponding homogeneous equation is $5u'' + 50u' + 490u = 0$, and its characteristic 
equation is $5r^2 + 50r + 490 = 0$. The roots of the characteristic equation are $r_{1,2} = -5 \pm i \sqrt{73}$ 
($\sqrt{73} \approx 8.54404$) and the complementary solution (or general solution of the corresponding 
homogeneous equation) is

$$u_c(t) = c_1 e^{-5t} \cos(\sqrt{73} t) + c_2 e^{-5t} \sin(\sqrt{73} t).$$

A particular solution for the nonhomogeneous equation can be found using the method of 
undetermined coefficients: the correct form for a particular solution is

$$U(t) = A \cos(t/2) + B \sin(t/2),$$

(with "s = 0") and substituting this expression into the nonhomogeneous equation and 
collecting coefficients allows us to determine them as

$$A = -\frac{160}{153,281} (\approx -0.0010438), \quad B = \frac{3,128}{153,281} (\approx 0.020407),$$

so the particular solution is

$$U(t) = -\frac{160}{153,281} \cos(t/2) + \frac{3,128}{153,281} \sin(t/2).$$

Then the general solution of the nonhomogeneous differential equation is

$$u(t) = c_1 e^{-5t} \cos(\sqrt{73} t) + c_2 e^{-5t} \sin(\sqrt{73} t) + \frac{1}{153,281} [-160 \cos(t/2) + 3128 \sin(t/2)].$$

Using the initial conditions

$$u(0) = 0, \quad u'(0) = \frac{3}{100}$$

gives

$$c_1 = \frac{160}{153,281} (\approx 0.0010438), \quad c_2 = \frac{383,443 \sqrt{73}}{1,118,951,300} (\approx 0.0029279).$$
Thus the position of the mass at any time $t$ is given exactly by
\[ u(t) = \frac{1}{153,281} \left[ 160 e^{-5t} \cos(\sqrt{73}t) + \frac{363,443 \sqrt{73}}{7,300} e^{-5t} \sin(\sqrt{73}t) - 160 \cos(t/2) + 3,128 \sin(t/2) \right] \]

or approximately by
\[ u(t) = 0.0010438 e^{-5t} \cos(8.544004t) + 0.0029279 e^{-5t} \sin(8.544004t) \\
- 0.0010438 \cos(0.5t) + 0.020407 \sin(0.5t). \]

(b) The transient part is the complementary solution $u_c(t)$ (since $u_c(t) \to 0$ as $t \to \infty$) and the steady-state part (or the forced response) is the particular solution $U(t)$, which does not approach 0 as $t \to \infty$.

(c) It is easiest to use the formula from the textbook or lecture notes: the maximum possible amplitude of the forced response is attained when the forcing frequency $\omega$ has the value
\[ \omega_{\text{max}} = \omega_0 \sqrt{1 - \frac{\gamma^2}{2\eta k}} = \sqrt{\frac{k}{m}} \sqrt{1 - \frac{\gamma^2}{2\eta k}} = \sqrt{\frac{490}{5}} \sqrt{1 - \frac{80^2}{2 \times 490}} = 4\sqrt{3} \approx 6.928203 \text{ s}^{-1}. \]

(Alternatively, you could find the particular solution, or forced response $u = U(t)$ for
\[ 5u'' + 50u' + 490u = 10 \sin(\omega t) \]
as
\[ U(t) = A(\omega) \cos(\omega t) + B(\omega) \sin(\omega t), \]
find the undetermined coefficients $A(\omega)$ and $B(\omega)$ as expressions that depend on $\omega$, then use first-year calculus to find the critical point $\omega = \omega_{\text{max}}$, of the function
\[ R(\omega) = \sqrt{A(\omega)^2 + B(\omega)^2}, \]
and verify that $R(\omega)$ attains its maximum value at the critical point.)

4. An undamped mass-spring system with a mass that weighs 6 lb and a spring constant of 12 lb/ft is suddenly set in motion at $t = 0$ by an external force of $4 \cos(7t)$ lb. Determine the displacement of the mass at any time $t \geq 0$, and write the displacement as a product of two trigonometric functions of different frequencies. Sketch the graph of the position versus $t$, showing the amplitude modulation with the slow frequency.

[12] Solution: In FPS units the weight is $mg = 6$ lb, so the mass is $m = (mg)/g = (6 \text{ lb})/(32 \text{ ft/s}^2) = \frac{3}{16}$ slug. The spring constant is given as $k = 12 \text{ lb/ft}$. We interpret the words “suddenly set in motion” as meaning that the initial position is at equilibrium, at rest: $u(0) = 0$, $u'(0) = 0$. The differential equation to solve is
\[ \frac{3}{16} u'' + 12u = 4 \cos(7t), \]
with initial conditions $u(0) = 0$, $u'(0) = 0$. This is a linear second-order differential equation with constant coefficients, and it is nonhomogeneous. The corresponding homogeneous
equation is \( \frac{3}{16} u'' + 12 u = 0 \), its characteristic equation \( \frac{3}{16} r^2 + 12 = 0 \) has purely imaginary roots \( r_{1,2} = \pm i \sqrt{16 \cdot 12/3} = \pm i 8 \), and the complementary solution is

\[ u_c(t) = c_1 \cos(8t) + c_2 \sin(8t), \]

where \( c_1 \), \( c_2 \) are arbitrary constants. Since the natural frequency \( \omega_0 = 8 \) and the forcing frequency \( \omega = 7 \) are not equal, a particular solution for the nonhomogeneous equation (by the method of undetermined coefficients) has the form

\[ U(t) = A \cos(7t) + B \sin(7t), \]

for some coefficients \( A \) and \( B \). Substituting \( u = U(t) \) into the nonhomogeneous equation leads to \( A = \frac{64}{45} (\approx 1.42222) \), \( B = 0 \), so we have

\[ U(t) = \frac{64}{45} \cos(7t), \]

and the general solution of the nonhomogeneous equation is

\[ u(t) = c_1 \cos(8t) + c_2 \sin(8t) + \frac{64}{45} \cos(7t), \]

where \( c_1 \) and \( c_2 \) are arbitrary constants. Using the initial conditions

\[ u(0) = 0, \quad u'(0) = 0 \]

gives

\[ c_1 = -\frac{64}{45}, \quad c_2 = 0, \]

and the position of the mass at any time \( t \) is given by

\[ u(t) = -\frac{64}{45} \cos(8t) + \frac{64}{45} \cos(7t). \]

Then using the trigonometric identities

\[ \cos(A + B) = \cos(A) \cos(B) - \sin(A) \sin(B), \quad \cos(A - B) = \cos(A) \cos(B) + \sin(A) \sin(B), \]

with

\[ A = \frac{1}{2}(\omega_0 + \omega)t = \frac{1}{2}(8t + 7t) = \frac{15}{2} t, \quad B = \frac{1}{2}(\omega_0 - \omega)t = \frac{1}{2}(8t - 7t) = \frac{1}{2} t, \]

so that

\[ A + B = 8t, \quad A - B = 7t, \]

we get

\[ u(t) = \frac{128}{45} \sin \left( \frac{1}{2} t \right) \sin \left( \frac{15}{2} t \right). \]

This can be viewed as a sinusoidal signal \( \sin \left( \frac{15}{2} t \right) \) vibrating with a "fast" frequency \( \frac{15}{2} \), with a modulated amplitude \( \frac{128}{45} \sin \left( \frac{1}{2} t \right) \) that vibrates at a "slow" frequency \( \frac{1}{2} \).
Figure 1: Graph of $u(t)$ versus $t$ for Question 4.