The University of British Columbia
4 October 2018
Midterm for MATH 104, Section 101
Instructor: Senping Luo

Closed book examination

Last Name ___________________________ First ___________________________

Signature ____________________________ Solutions

Student Number ________________________

No memory aids are allowed. Simple calculators are allowed. No communication or other electronic devices. Show all your work; little or no credit will be given for a numerical answer without the correct accompanying work. If you need more space than the space provided, use the back of the previous page. Where boxes are provided for answers, put your final answers in them.

Midterms written in pencil will not be considered for regrading.

Rules governing examinations

- Each candidate must be prepared to produce, upon request, a UBCCard for identification.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
  - (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners.
  - (b) Speaking or communicating with other candidates.
  - (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
- Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

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1. **Short Answer Questions.** Each question is worth 3 points. Put your final answer in the box provided, but NO CREDIT will be given for the answer without the correct accompanying work.

(a) Evaluate
\[
\lim_{{x \to 0}} \frac{\sin(5x) \sin(3x)}{\sin(2x) \sin(4x)}.
\]

\[
= \lim_{{x \to 0}} \frac{\sin(5x)}{\sin(2x)} \cdot \frac{\sin(3x)}{\sin(4x)}
\]

\[
= \frac{5}{2} \cdot \frac{3}{4} = \frac{15}{8}.
\]

**Answer:** \[
\frac{15}{8}
\]

(b) Calculate \( f'(x) \) where \( f(x) = \frac{e^x - 3x}{1 + x} \).

\[
f'(x) = \frac{(e^x - 3x)'(1 + x) - (e^x - 3x)(1 + x)'}{(1 + x)^2}
\]

\[
= \frac{e^x (1 + x) - (e^x - 3x)}{(1 + x)^2}
\]

\[
= \frac{xe^x - 3}{(1 + x)^2}.
\]

**Answer:** \[
\frac{xe^x - 3}{(1 + x)^2}
\]

c) Evaluate
\[
\lim_{{t \to 0}} \frac{\sqrt{2 + t^2} - \sqrt{2}}{t^2}.
\]

\[
= \lim_{{t \to 0}} \frac{(\sqrt{2 + t^2} - \sqrt{2})(\sqrt{2 + t^2} + \sqrt{2})}{t^2 (\sqrt{2 + t^2} + \sqrt{2})}
\]

\[
= \lim_{{t \to 0}} \frac{2 + t^2 - 2}{t^2 (\sqrt{2 + t^2} + \sqrt{2})}
\]

\[
= \lim_{{t \to 0}} \frac{1}{\sqrt{2 + t^2} + \sqrt{2}} = \frac{1}{2\sqrt{2}}.
\]

**Answer:** \[
\frac{1}{2\sqrt{2}}
\]
(d) Let $f(x) = \sqrt{4-x^2}$ for $-2 \leq x \leq 2$. Find the equation of the tangent line to $f(x)$ at $x = \sqrt{3}$.

\[
\begin{align*}
  f(x) &= (4-x^2)^{\frac{1}{2}}; \\
  f'(x) &= \frac{1}{2} (4-x^2)^{-\frac{1}{2}} \cdot (4-x^2); \\
  &= \frac{1}{\sqrt{4-x^2}} \\
  \Rightarrow \quad f'(\sqrt{3}) &= -\sqrt{3}, \quad f(\sqrt{3}) = 1 \\
\end{align*}
\]

Point-slope form: $y - f(\sqrt{3}) = f'(\sqrt{3})(x - \sqrt{3})$

\[
y + \sqrt{3} (x - \sqrt{3}) \quad \text{or} \quad y = -\sqrt{3} x + 4
\]

(e) Find the values of a and b so that $f(x) = \begin{cases} 2 + 3e^x & \text{if } x < 0, \\ ax + b & \text{if } x \geq 0 \end{cases}$ is differentiable everywhere.

The functions $2 + 3e^x$ and $ax + b$ are differentiable on their domain. For the piecewise function $f(x)$, we only need to consider the point $x = 0$.

Continuous at $x = 0$ implies

\[
\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) \quad \Rightarrow \quad 2 + 3e^0 = a \cdot 0 + b \quad (1)
\]

and differentiable at $x = 0$ implies

\[
\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^+} f'(x) \quad \Rightarrow \quad (2 + 3e^x)'\bigg|_{x=0} = (ax + b)'\bigg|_{x=0} \\
\Rightarrow \quad 3e^0 = a \quad (2)
\]

Solving (1), (2), we have $a = 3$, $b = 5$.
2. Definition of the Derivative.

(a) Carefully state the definition of the derivative of a function \( f(x) \) at a point \( x = a \).

The derivative of \( f(x) \) at \( x = a \) is the limit of the difference quotients (if it exists):

\[
\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}
\]

(b) Use the definition of the derivative to compute the derivative of \( f(x) = 3 - \sqrt{x+2} \) at \( x = 2 \). NO CREDIT will be given for using any other method.

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
= \lim_{h \to 0} \frac{(3 - \sqrt{x+h+2}) - (3 - \sqrt{x+2})}{h}
= \lim_{h \to 0} \frac{\sqrt{x+2} - \sqrt{x+h+2}}{h} = \frac{-1}{\sqrt{x+2} + \sqrt{x+h+2}}
\]

(c) Calculate the derivative of \( f(x) = 3 - \sqrt{x+2} \) at \( x = 2 \) by differential rules. In this way, you can check the answer in (b).

\[
f'(x) = \frac{d}{dx} \left( 3 - (x+2)^{\frac{1}{2}} \right) = -\frac{1}{2} (x+2)^{-\frac{1}{2}}
= -\frac{1}{2 \sqrt{x+2}}
\]

\[
f'(2) = -\frac{1}{4}
\]

Remark: We have two ways to calculate the derivative of a function. One is by definition (like (b)), another is by differential rules (product, quotient, chain rule, like (c)).

These two ways should give you the same answer since they calculate the same thing (derivative). You may also realize the differential rules greatly simplify the process of derivative. This is why we learn these rules.
[10] 3 Application. The cost of producing \( q \) tons of sugar is given by \( C(q) = \frac{q^2}{10} + 8q + 150 \). When 10 tons of sugar are sold, the price at which the producer can sell them is \$280. For every extra 10 tons of sugar the producer sells in the market, the price drops by \$1 per ton.

(a) [2] Find the linear demand equation for sugar. Use \( p \) for the unit price and \( q \) for the weekly demand.
(b) [2] Find the weekly revenue function \( R(q) \) as a function of \( q \).
(c) [2] Find the break-even points for the sugar producer. Give both the price \( p \) and quantity \( q \) at each of these points.
(d) [2] Find the derivative of the profit function, \( P'(q) \). (This derivative is usually called the marginal profit.)
(e) [2] Suppose that the producer is producing and selling \( \hat{q} \) tons of sugar, where \( \hat{q} \) corresponds to the largest \( q \)-value of all the break-even points. Should the producer increase or decrease the amount of sugar it is producing to increase its profit? Explain your answer.

\[(a) \quad q \geq 290 - p \]

\[(b) \quad R(q) = pq = (290 - q)q \]

\[(c) \quad R(q) = C(q) \quad \Rightarrow \quad \frac{11}{10}q^2 - 282q + 150 = 0 \]

\[= \begin{cases} q \approx 255.54 \quad \text{or} \quad q \approx 34.46 \end{cases} \]

\[(d) \quad P(q) = R(q) - C(q) = -\frac{11}{10}q^2 + 282q - 150 \]

\[P(q) = -\frac{11}{5}q + 282 \]

\[(e) \quad \text{\underline{Decrease}}\]
4. Tangent line and zeros existence theorem.

(a) Consider the function \( f(x) = 1 - 2x - x^2 \). Find the point \((x_0, y_0)\) such that the tangent line to \( f \) at \((x_0, y_0)\) is parallel to the line \( y = x + 10 \).

Parallel implies slopes equal

\[
\Rightarrow \quad f'(x_0) = 1, \quad \Rightarrow \quad -2 - 2x_0 = 1
\]

\[
\Rightarrow \quad x_0 = -\frac{3}{2}, \quad y_0 = f(x_0) = \frac{7}{4}
\]

\[
\Rightarrow \quad (x_0, y_0) = \left(-\frac{3}{2}, \frac{7}{4}\right)
\]

(b) Show that \( \cos(x) = x^3 \) has a solution.

Let \( f(x) = \cos(x) - x^3 \), then \( f(x) \) is a continuous function on \((-\infty, +\infty)\). Next,

\[
f(0) = \cos(0) - 0^3 = 1 > 0
\]

\[
f(1) = \cos(1) - 1^3 < 0
\]

\( f(0) f(1) < 0 \), hence by zeros existence theorem, there exists a \( c \in (0, 1) \) such that \( f(c) = 0 \), that is \( \cos(x) = x^3 \) has a solution \( c \in (0, 1) \).
5. Bonus problems. The total marks for this midterm is 50, notice that the previous problems already worth 50. For the following problem you can still get marks. But if you get more than 50, you will still get 50 since the full marks is 50.

(a) [3] How do you connect the concept 'marginal' in economic with 'derivative' in mathematics?

(b) [2] How do you understand \( \lim_{x \to 0} \frac{\sin(x)}{x} = 1 \)?

(c) [3] What's the difference between \( f'(g(x)) \) and \( [f(g(x))]' \)? What about \( \frac{d}{du} f(u)|_{u=g(x)} \) and \( \frac{d}{dx} f(g(x)) \)? Give an example to illustrate your point.