HOMEWORK 5: OPTIMIZATION, INEQUALITY; MONOTONICITY AND CONCAVITY

1: A-G-M inequality ($n = 2, 3$ and general).
   (1) Consider the function
   \[ f(x) = \left(\frac{a + x}{2}\right)^2 - ax. \]
   Here $a \geq 0$ is a constant. Find the minimum of $f(x)$ on the interval $[0, +\infty)$.
   (2) Consider the function
   \[ f(x) = \left(\frac{a + b + x}{3}\right)^3 - abx. \]
   Here $a, b \geq 0$ are constants. Find the minimum of $f(x)$ on the interval $[0, +\infty)$.
   (3) What can you conclude from (1), (2) ?
   (4) [Optional] Consider the function
   \[ f(x) = (\frac{a_1 + \cdots + a_{n-1} + x}{n})^n - a_1 \cdots a_{n-1}x. \]
   Here $a_1, \cdots, a_{n-1} \geq 0$. Find the minimum of $f(x)$ on the interval $[0, +\infty)$.

2: Design a cylindrical can of volume $900\text{cm}^3$ so that it uses the least amount of metal. In other words, minimize the surface area of the can (including its top and bottom).

3: Find the critical points, local extremes, inflection points and sketch the graph of
   • $f(x) = \frac{1}{7}x^3 - \frac{1}{7}x^2 - 2x + 3$.
   • $f(x) = 3x^4 - 8x^3 + 6x^2 + 1$.
   • $f(x) = \cos(x) + \frac{1}{2}x$ over $[0, \pi]$.
   You can check your answer by graphing on the computer.