HOMEWORK 2: ZEROS EXISTENCE THEOREM AND DEFINITION OF DERIVATIVE

1: Evaluate the following limits:

1) \( \lim_{x \to 0} \frac{\sin(5x) \sin(3x)}{\sin(4x) \sin(2x)} \)
2) \( \lim_{x \to 1} \left( \frac{1}{1-x} - \frac{2}{1-x^2} \right) \)
3) \( \lim_{h \to 0} \frac{2(a+h)^2 - 2a^2}{h} \)
4) \( \lim_{s \to 0} \frac{1 - \sqrt{s^2 + 1}}{s^2} \).

2: [Application of zeros existence Theorem] 1) Show that \( \cos(x) = x \) has a solution in the interval \([0, 1]\). 2) Show that \( g(t) = t^2 \tan(t) \) takes value \( \frac{1}{2} \) for some \( t \) in \([0, \frac{\pi}{4}]\).

3: [Definition of derivative] 1) Carefully state the definition of the derivative of a function \( f(x) \) at a point \( x = a \). 2) Use the definition of the derivative to compute the derivative of \( f(x) = 3 - \sqrt{x + 4} \) at \( x = 6 \).

4: Find a constant \( b \) such that \( h(x) \) is continuous at \( x = 4 \), where

\[
  h(x) = \begin{cases} 
    x^2 - 1 & \text{if } x < 4, \\
    b - x^3 + \sqrt{x} & \text{if } x \geq 4.
  \end{cases}
\]

[Hint: use the left hand limit and right hand limit]. With this choice of \( b \), please determine the derivative of \( h(x) \) at \( x = 4 \) exists or not. [Hint: you may use \((x^a)' = ax^{a-1}\)].