APPLICATION 5: OPTIMIZATION

1: A-G-M inequality and perimeter, area on the plane.
(1) Show that $\ln(x)$ is concave down on $(0, +\infty)$.
(2) Use (1) prove the A-G-M inequality,
$$\sqrt[n]{x_1x_2\cdots x_n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n}, \text{ where } x_1, x_2, \ldots, x_n \geq 0,$$
" = " holds iff $x_1 = x_2 = \cdots = x_n$.
(3) Consider the function
$$f(x) = \left(\frac{a_1 + \cdots + a_{n-1} + x}{n}\right)^n - a_1 \cdots a_{n-1}x.$$ 
Here $a_1, \cdots a_{n-1} \geq 0$. Find the minimum of $f(x)$ on the interval $[0, +\infty)$.
(4) Use (3) prove A-G-M inequality in (2) again.
(5) Show the iso-perimeter inequality for triangle.
$$S \leq \frac{p^2}{2\sqrt{3}},$$
where $S$ is the area and $p$ is half of the perimeter, " = " holds when $a = b = c$.
(6) Show that among all triangles with given perimeter, the equilateral triangle has the maximum area.
(7) Show that among all triangles with given area, the equilateral triangle has the least perimeter.

Solution of (5). Assume that perimeter is $D(= a + b + c = 2p)$, then the area of the triangle is
$$S = \sqrt{p(p-a)(p-b)(p-c)}, \quad p = \frac{a+b+c}{2}.$$ 

By A-G-M inequality,
$$(p-a)(p-b)(p-c) \leq \frac{(a+b+p-b+p-c)^3}{3} = \frac{(3p-(a+b+c))^3}{3} = \frac{p^3}{3},$$
and " = " holds if $p-a = p-b = p-c$, or $a = b = c$, or maximum achieved at equilateral triangle.

For the area function $S$, it achieves its maximum at the same point as $(p-a)(p-b)(p-c)$, i.e.,
at the same point, $a = b = c$. That is
$$S = \sqrt{p(p-a)(p-b)(p-c)} = \sqrt{p\sqrt{(p-a)(p-b)(p-c)}}$$
$$\leq \sqrt{p\sqrt{(\frac{p}{3})^3}} = \sqrt{\frac{p^3}{3}},$$
with " = " holds $a = b = c$.

2: Your task is to build a road joining a ranch to a highway that enables drivers to reach the city in the shortest time. How should this be done if the speed limit is 60km/h on the road and 110km/h on the highway? The perpendicular distance from the ranch to the highway is 30km, and the city is 50km down the highway.

General steps: 1: Choose variable. 2: Find the objective function and interval. 3: Optimize. $T(x) = \frac{\sqrt{30^2+x^2}}{60} + \frac{50-x}{110}$.

3: Design a cylindrical can of volume $900cm^3$ so that it uses the least amount of metal. In other words, minimize the surface area of the can(including its top and bottom).
$$A = 2\pi r^2 + \frac{1800}{r}, \quad r_0 = \left(\frac{450}{\pi}\right)^{1/3} \approx 5.23, \quad h = 2(\frac{450}{\pi})r^{-2} = 2(\frac{450}{\pi})^{1/3} \approx 10.46cm.$$
