Lecture Notes on Topics in Complex Geometry

Sébastien Picard

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Contents

1 Complex Geometry

This section is an introduction to complex geometry. For other references in the style of these notes, see Kodaira's book [19], Chapter 1 of Siu's notes [24], Chapter 1 of Song-Weinkove's notes [25], or Chapter 1 of Szekelyhidi's book [26].

1.1 Complex manifolds

Let $\Omega \subseteq \mathbb{C}^n$ be a domain. We denote complex variables by

$$z = (z^1, \dots, z^n),$$

so that

$$z^k = x^k + iy^k, \quad \bar{z}^k = x^k - iy^k, \quad k \in \{1, \dots, n\},$$

and the real variables on $\Omega \subseteq \mathbb{R}^{2n}$ are $(x^1, y^1, \dots, x^n, y^n)$. The inverse transformation is

$$x^k = \frac{1}{2}(z^k + \bar{z}^k), \quad y^k = \frac{1}{2i}(z^k - \bar{z}^k). \tag{1.1}$$

By the chain rule, for a smooth function $f : \Omega \to \mathbb{C}$, we have

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)f, \quad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)f. \tag{1.2}$$

Definition 1.1. *Let $f : \Omega \to \mathbb{C}^k$ be a C^1 function with components*

$$f = (f^1(p), \dots, f^k(p)).$$

We say f is holomorphic if

$$\frac{\partial f^i}{\partial \bar{z}^k} = 0$$

for all i, k.

Let M be a compact complex manifold. This means that M is a smooth compact manifold admiting a finite cover by open sets $M = \bigcup_i U_i$ with homeomorphisms

$$z_U : U \to \mathcal{U} \subseteq \mathbb{C}^n, \quad z_U(p) = (z_U^1(p), \dots, z_U^n(p))$$

with the following property. For any pair $(U, z_U), (V, z_V)$ with $U \cap V \neq \emptyset$, then we can write

$$z_V^p = f_{VU}{}^p(z_U), \quad f_{VU} = z_V \circ z_U^{-1}$$

with $f_{VU} : z_U(U \cap V) \to z_V(U \cap V)$ a holomorphic bijective function with holomorphic inverse.

Example 1.2. $\mathbb{P}^1 = (\mathbb{C}^2 \backslash \{0\}) / \sim$, where points $p \in \mathbb{P}^1$ are written as

$$p = [Z_0 : Z_1]$$

and $[Z_0 : Z_1] \sim [X_0 : X_1]$ if and only if $(Z_0, Z_1) = \lambda(X_0, X_1)$ for $\lambda \in \mathbb{C}^*$. Holomorphic charts are:

- $U_0 = \{Z_0 \neq 0\}$, with coordinate
$$z = \frac{Z_1}{Z_0}.$$
- $U_1 = \{Z_1 \neq 0\}$, with coordinate
$$\tilde{z} = \frac{Z_0}{Z_1}.$$

The change of coordinates function $f_{10} : z(U_0 \cap U_1) \to \tilde{z}(U_0 \cap U_1)$ is

$$f_{10} : \mathbb{C}^* \to \mathbb{C}^*, \quad \tilde{z} = f_{10}(z) = \frac{1}{z}.$$

Example 1.3. Complex projective space in higher dimensions $\mathbb{P}^n = (\mathbb{C}^{n+1} \backslash \{0\}) / \sim$ is defined similarly. Points are denoted

$$p = [Z_0 : Z_1 : \cdots : Z_n]$$

and $[Z_0 : Z_1 : \cdots : Z_n] \sim [X_0 : X_1 : \cdots : X_n]$ if and only if $(Z_0, Z_1, \dots, Z_n) = \lambda(X_0, X_1, \dots, X_n)$ for $\lambda \in \mathbb{C}^*$. Holomorphic charts are of the form $U_k = \{Z_k \neq 0\}$. For example, (U_0, z) has coordinate

$$z = (z^1, \dots, z^n) = \left(\frac{Z_1}{Z_0}, \dots, \frac{Z_n}{Z_0} \right),$$

while $(U_1, \tilde{z})$ has coordinate

$$\tilde{z} = (\tilde{z}^1, \dots, \tilde{z}^n) = \left(\frac{Z_0}{Z_1}, \frac{Z_2}{Z_1}, \dots, \frac{Z_n}{Z_1} \right),$$

and the change of coordinates function is $\tilde{z} = f_{10}(z)$ with

$$\tilde{z}^1 = \frac{1}{z^1}, \quad \tilde{z}^k = \frac{z^k}{z^1}, \quad k \in \{2, \dots, n\}.$$

The coordinates on the other open sets U_k are defined similarly.

Example 1.4. Let $P(Z_0, \dots, Z_n)$ be a homogeneous polynomial of degree r, meaning $P(\lambda Z_0, \dots, \lambda Z_n) = \lambda^r P(Z_0, \dots, Z_n)$. Suppose that P has the property that only the point $Z_0 = \cdots = Z_n = 0$ solves $\frac{\partial P}{\partial Z_i} = 0$ for all i. Then

$$X = \{x \in \mathbb{P}^n : P(x) = 0\}$$

defines a complex manifold of dimension $n-1$. To see this, we look at $\{P = 0\}$ inside the local charts $U_k \subseteq \mathbb{P}^n$. For example, in coordinates (U_0, z), the equation defining X is

$$0 = f(z^1, \dots, z^n) = P(1, z^1, \dots, z^n).$$

At a point $p \in \{f = 0\}$, there must be a coordinate z^i such that $\frac{\partial f}{\partial z^i} \neq 0$, in other words one of $\frac{\partial P}{\partial Z_1}, \dots \frac{\partial P}{\partial Z_n}$ must be nonzero. The assumption on P is that we cannot have $\frac{\partial P}{\partial Z_i} = 0$ for all $i \in \{0, \dots, n\}$, so we only need to rule out $\frac{\partial P}{\partial Z_0} \neq 0$ with all other partials zero. This is ruled out by Euler's identity

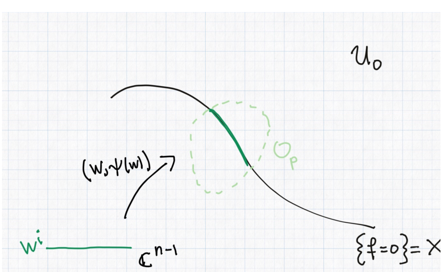
$$\sum_{i=0}^{n} Z_i \frac{\partial P}{\partial Z_i} = rP,$$

which shows that on $\{P = 0\} \cap U_0$ if $\frac{\partial P}{\partial Z_0} \neq 0$ then one of $\frac{\partial P}{\partial Z_i}$ for $i \geqslant 1$ must be non-vanishing. Euler's identity follows from $\frac{d}{dt}\big|_{t=1} P(tZ) = \frac{d}{dt}\big|_{t=1} t^r P(Z)$.

Without loss of generality, $\frac{\partial f}{\partial z^n} \neq 0$. By the holomorphic implicit function theorem, there exists a holomorphic function $\psi : V \to O_p$, $V \subset \mathbb{C}^{n-1}$, $p \in O_p \subset U_0$, such that

$$X \cap O_p = \{(w^1, \dots, w^{n-1}, \psi(w)) : w \in V\}.$$

This defines a coordinate chart (V, w).



There are two types of coordinate change:

- Suppose $p \in X \cap U_0$ satisfies both $\frac{\partial f}{\partial z^1} \neq 0$ and $\frac{\partial f}{\partial z^n} \neq 0$. Then we can parametrize X in a neighborhood of p by either (implicit function on $\frac{\partial f}{\partial z^n} \neq 0$)

$$(w^1, \dots, w^{n-1}, \psi(w))$$

or (implicit function on $\frac{\partial f}{\partial z^1} \neq 0$)

$$(\tilde{\psi}(\tilde{w}), \tilde{w}^1, \ldots, \tilde{w}^{n-1}).$$

The change of coordinates is

$$\tilde{w}^1 = w^2, \quad \tilde{w}^2 = w^3, \quad \ldots \quad \tilde{w}^{n-1} = \psi(w),$$

which is holomorphic.

• Suppose $p \in X \cap (U_0 \cap U_1)$. Then we can use coordinates coming from either U_0 or U_1. As before we denote (U_0, z) and $(U_1, \tilde{z})$ with $\tilde{z}^1 = 1/z^1$ and $\tilde{z}^k = z^k/z^1$ for $k \geqslant 2$. The submanifold appears as the equation

$$0 = P(1, z^1, z^2, \ldots, z^n)$$

on U_0, and

$$0 = P(\tilde{z}^1, 1, \tilde{z}^2, \ldots, \tilde{z}^n)$$

on U_1. Suppose $\frac{\partial P}{\partial Z_n} \neq 0$. As before, the implicit function theorem gives coordinates w^i and $\tilde{w}^i$, so that the equations become

$$0 = P(1, w^1, w^2, \ldots, \psi(w))$$

on U_0, and

$$0 = P(\tilde{w}^1, 1, \tilde{w}^2, \ldots, \tilde{\psi}(\tilde{w}))$$

on U_1. The change of coordinates is

$$\tilde{w}^1 = 1/w^1, \quad \tilde{w}^2 = w^2/w^1, \quad \ldots \quad \tilde{w}^{n-1} = w^{n-1}/w^1.$$

Other situations when e.g. $\frac{\partial P}{\partial Z_0} \neq 0$ and $\frac{\partial P}{\partial Z_1} \neq 0$ can also be worked out in a similar way.

1.2 Holomorphic vector bundles

1.2.1 Definitions and notation

We recall the cocycle definition of a rank r complex vector bundle. Let $M = \bigcup_i U_i$ be a finite covering of open coordinate charts, together with matrix-valued functions on the nonzero overlaps

$$t_{UV} : U \cap V \to GL(r, \mathbb{C})$$

satisfying

$$t_{UV}(p) = t_{UW}(p) t_{WV}(p), \quad p \in U \cap V \cap W.$$

We call the t_{UV} transition functions. They satisfy $t_{UU} = I_{r\times r}$ and $t_{UV}^{-1} = t_{VU}$. We define a complex vector bundle E by

$$E = \left(\bigcup_i U_i \times \mathbb{C}^r \right) / \sim,$$

where the relation is as follows. For $(p, u) \in U \times \mathbb{C}^r$ and $(p, v) \in V \times \mathbb{C}^r$, we identify $(p, u) \sim (p, v)$ if

$$u = t_{UV}(p) v.$$

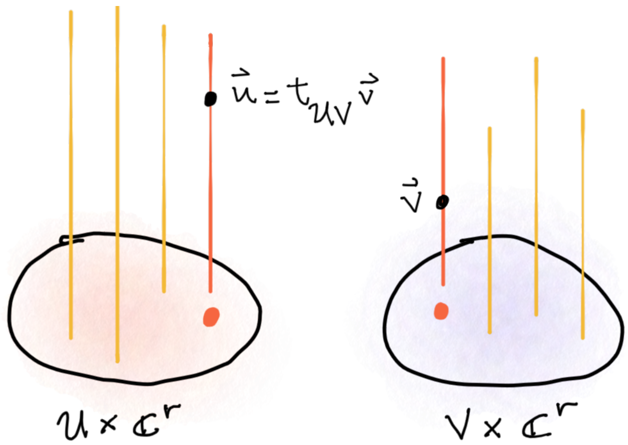
This is written using matrix notation. In terms of components, this is written

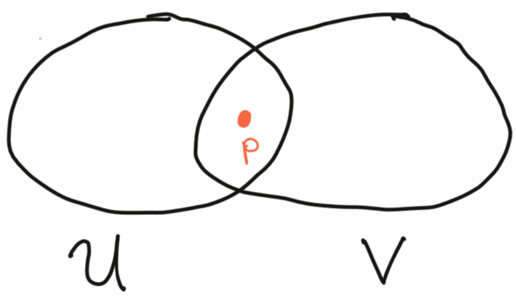
$$u^k = [t_{UV}(p)]^k{}_\ell v^\ell,$$

where repeated indices are summed. Here we write $v = (v^1, \ldots, v^r)$ and the components of the matrix t_{UV} are denoted ${t_{UV}}^i{}_j$, e.g. for 2x2,

$$\begin{bmatrix} u^1 \\ u^2 \end{bmatrix} = \begin{bmatrix} {t_{UV}}^1{}_1 & {t_{UV}}^1{}_2 \\ {t_{UV}}^2{}_1 & {t_{UV}}^2{}_2 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix}.$$

The $U_i \times \mathbb{C}^r$ are the trivializations of the bundle. The projection map $\pi : E \to M$ is given by $\pi(p, u) = p$.





We denote $E|_p = \pi^{-1}(p)$ to be the fiber over p, and note that $E|_p$ is a vector space of dimension r. For two points (p, u), (p, v) in the same trivialization $U_i \times \mathbb{C}^r$, the vector space structure is

$$a(p, u) + b(p, v) = (p, au + bv),$$

and one can check that this is well-defined.

• Note: we will call a complex vector bundle of rank 1 a line bundle.

Definition 1.5. *A complex vector bundle $\pi : E \to M$ over a complex manifold is holomorphic if the transition functions t_{UV} are holomorphic.*

• Note: taking U, V to be coordinate charts, then $U \cap V$ is viewed as an open set in $\mathbb{C}^n$. That $t_{UV} : U \cap V \to GL(r, \mathbb{C})$ is holomorphic means that each entry of the matrix is a holomorphic function of $p \in U \cap V \subseteq \mathbb{C}^n$.

Example 1.6. We will denote the trivial holomorphic line bundle by $\mathcal{O}_X \to X$. This means that the transition functions are $t_{UV} = 1$.

Definition 1.7. *Let $E \to M$ be a rank r complex vector bundle with trivializations U_i. A smooth section, denoted $s \in \Gamma(E)$, is given by local vector-valued smooth functions $\{U_i, s_{U_i}\}$ with $s_U : U \to \mathbb{C}^r$ satisfying*

$$s_U = t_{UV} s_V.$$

A section $s \in \Gamma(E)$ defines a well-defined map $s : M \to E$ such that

$$s(p) \in E|_p.$$

Indeed, in this formalism we set $s(p) = (p, s_U(p))$ when $p \in U$, and the condition $s_U = t_{UV} s_V$ ensures that if $p \in U \cap V$ then $(p, s_U(p)) \sim (p, s_V(p))$.

Remark 1.8. If $E \to M$ is a holomorphic bundle and the s_U are holomorphic functions, then we say s is a holomorphic section and write $s \in H^0(M, E)$.

Remark 1.9. In components, the transformation law is

$$s_U^k = {t_{UV}}^k{}_\ell s_V^\ell \tag{1.3}$$

e.g. for 3x3, this notation means

$$\begin{bmatrix} s_U^1 \\ s_U^2 \\ s_U^3 \end{bmatrix} = \begin{bmatrix} {t_{UV}}^1{}_1 & {t_{UV}}^1{}_2 & {t_{UV}}^1{}_3 \\ {t_{UV}}^2{}_1 & {t_{UV}}^2{}_2 & {t_{UV}}^2{}_3 \\ {t_{UV}}^3{}_1 & {t_{UV}}^3{}_2 & {t_{UV}}^3{}_3 \end{bmatrix} \begin{bmatrix} s_V^1 \\ s_V^2 \\ s_V^3 \end{bmatrix},$$

on $U \cap V$.

There is another viewpoint on this from the perspective of basis vectors rather than vector components. For a trivialization $U \times \mathbb{C}^r$, let

$$e_a^U(p) = (p, (0, \dots, 0, 1, 0, \dots, 0)) \in U \times \mathbb{C}^r,$$

where the 1 is at the ath position. Then $\{e_1^U(p), \dots e_r^U(p)\}$ is a basis for $E|_p$, and we say that $\{e_a^U\}$ is a local frame over U.

On an overlap $U \cap V$, the same basis vector can be written in two different ways. We note that $e_a^U \sim e_b^V {t_{VU}}^b{}_a$. Instead of the $\sim$ notation, this is usually just written

$$e_a^U = e_b^V {t_{VU}}^b{}_a. \tag{1.4}$$

Here is how to see this in the 2×2 case and $a = 1$. By definition, $e_1^U \sim e_b^V {t_{VU}}^b{}_1$ if and only if

$$e_1^U = t_{UV} e_b^V {t_{VU}}^b{}_1.$$

which is

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = [t_{UV}] \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} {t_{VU}}^1{}_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} {t_{VU}}^2{}_1 \right) = [t_{UV}] \left([t_{VU}] \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

and this holds since $t_{UV} t_{VU} = I_{2\times 2}$.

For a section $s \in \Gamma(E)$, we will sometimes make the frame explicit and write

$$s = s^a e_a.$$

In the 2×2 case, this notation means

$$\begin{bmatrix} s^1(p) \\ s^2(p) \end{bmatrix} = s^1(p) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + s^2(p) \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Note that on $U \cap V$, then

$$s^a_U e^U_a = s^a_V e^V_a$$

so we simply write $s = s^a e_a$. Indeed, substituting the transformation laws (1.3), (1.4) gives

$$s^a_U e^U_a = [t_{UV}{}^a{}_b s^b_V][e^V_c t_{VU}{}^c{}_a] = \delta^c{}_b s^b_V e^V_c = s^b_V e^V_b.$$

In terms of linear algebra, this is just the statement that the same vector v will appear in different components v^a using different bases e_a.

Example 1.10. Let $M = \bigcup_i U_i$ be a complex manifold with holomorphic coordinate charts (U, z). The holomorphic tangent bundle $T^{1,0}M \to M$ is the holomorphic bundle defined by transition matrices

$$t_{UV}{}^k{}_i = \frac{\partial z^k_U}{\partial z^i_V}.$$

Sections $X \in \Gamma(T^{1,0}M)$ are denoted

$$X = X^p(z) \frac{\partial}{\partial z^p}.$$

On an overlap of coordinate charts (U, z), $(\tilde{U}, \tilde{z})$, components transform as

$$\tilde{X}^p = \frac{\partial \tilde{z}^p}{\partial z^\ell} X^\ell$$

while the basis transforms as

$$\frac{\partial}{\partial \tilde{z}^k} = \frac{\partial z^p}{\partial \tilde{z}^k} \frac{\partial}{\partial z^p}.$$

It follows that

$$X^p \frac{\partial}{\partial z^p} = \tilde{X}^p \frac{\partial}{\partial \tilde{z}^p}$$

on overlaps.

Let $E \to M$, $F \to M$ be two holomorphic vector bundles with transition functions t_{UV}, $\tilde{t}_{UV}$ with respect to a trivialization $M = \bigcup_i U_i$. An isomorphism of holomorphic bundles $h : E \to F$ is given by a collection $\{h_U : U \to GL(r, \mathbb{C})\}$ of holomorphic invertible matrices satisfying

$$h_U = \tilde{t}_{UV} h_V t^{-1}_{UV}. \tag{1.5}$$

This definition is such that h is a well-defined isomorphism from fibers of E to fibers of F.

$$h(p) : E|_p \to F|_p$$

This amounts to the statement that if $v \sim w$ in E, then $hv \sim hw$ in F. Indeed, if $v_U = t_{UV} w_V$ then

$$h_U v_U = (\tilde{t}_{UV} h_V t^{-1}_{UV})(t_{UV} w_V) = \tilde{t}_{UV}(h_V w_V).$$

Example 1.11. We return to the example $M = \mathbb{P}^1 = U_0 \cup U_1$. There are two charts (U_0, z), $(U_1, \tilde{z})$ and $\tilde{z} = z^{-1}$. Therefore

$$\frac{\partial}{\partial z} = \frac{\partial \tilde{z}}{\partial z}\frac{\partial}{\partial \tilde{z}} = -\frac{1}{z^2}\frac{\partial}{\partial \tilde{z}} = -\tilde{z}^2\frac{\partial}{\partial \tilde{z}}.$$

Said otherwise, a section of $T^{1,0}M$ may be written on $U_0 \cap U_1$ as $v(z)\frac{\partial}{\partial z}$ or $\tilde{v}(\tilde{z})\frac{\partial}{\partial \tilde{z}}$ with

$$\tilde{v} = -\tilde{z}^2 v.$$

so that the transition function is $t_{10} = -\tilde{z}^2$.

For example, defining $\frac{\partial}{\partial z}$ over U_0 extends to a global vector field V over M by setting $-\tilde{z}^2\frac{\partial}{\partial \tilde{z}}$ over U_1. However, even though $\frac{\partial}{\partial z}$ is nowhere vanishing over U_0, this vector field must acquire a zero at $\tilde{z} = 0$ in U_1. In component notation, this vector field V is given by the data

$$V = \{(U_0, v(z)), (U_1, \tilde{v}(\tilde{z}))\} \in H^0(M, T^{1,0}M)$$

with

$$v(z) = 1, \quad \tilde{v}(\tilde{z}) = -\tilde{z}^2.$$

Example 1.12. Let $k \in \mathbb{Z}$. Define the bundle $\mathcal{O}(k) \to \mathbb{P}^1$ with trivializations (U_0, z), $(U_1, \tilde{z})$ by setting $t_{10} = \tilde{z}^k$, so that sections transform as

$$\tilde{s} = \tilde{z}^k s.$$

The previous example, combined with (1.5) and suitable choice of h_{U_0}, h_{U_1}, shows that $T^{1,0}\mathbb{P}^1 \cong \mathcal{O}(2)$. Let $k > 0$.

• There are no holomorphic sections of $\mathcal{O}(-k)$. Suppose such a section appears as a holomorphic function $s(z)$ over the trivialization U_0. Then over U_1, that same section takes the form $\tilde{s} = \tilde{z}^{-k}s$, which in the $\tilde{z}$ coordinates belonging to U_1 is

$$\tilde{s}(\tilde{z}) = \tilde{z}^{-k}s(\tilde{z}^{-1}).$$

Writing $s(z) = \sum_{i=0}^{\infty} a_i z^i$, we see that $\tilde{s}(\tilde{z})$ must have a pole and cannot be holomorphic.

• Holomorphic sections of $\mathcal{O}(k)$ correspond to homogeneous polynomials $P(Z_0, Z_1)$ of degree k: any section $\sigma \in H^0(\mathbb{P}^1, \mathcal{O}(k))$ is $\sigma = \{(U_0, s), (U_1, \tilde{s})\}$ locally the form $s = P(Z_0, Z_1)/Z_0^k$ over U_0, and of the form $\tilde{s} = P(Z_0, Z_1)/Z_1^k$ over U_1. Indeed, let $\sigma \in H^0(\mathbb{P}^1, \mathcal{O}(k))$ be an arbitrary holomorphic section. Then $s(z)$, $\tilde{s}(\tilde{z})$ are both holomorphic and

$$\tilde{s}(\tilde{z}) = \tilde{z}^k s(\tilde{z}^{-1}). \tag{1.6}$$

After writing $\tilde{s} = \sum_{k=0}^{\infty} b_k \tilde{z}^k$, $s = \sum_{k=0}^{\infty} a_k z^k$ and comparing coefficients, we see that $s(z) = a_0 + a_1 z + \cdots + a_k z^k$. It follows that

$$s = \frac{1}{Z_0^k}\left[a_0 Z_0^k + a_1 Z_0^{k-1} Z_1 + \cdots + a_k Z_1^k\right]$$

since on $U_0 = \{Z_0 \neq 0\}$ the coordinate is $z = Z_1/Z_0$. The transformation (1.6) implies

$$\tilde{s} = \frac{1}{Z_1^k}\left[a_0 Z_0^k + a_1 Z_0^{k-1} Z_1 + \cdots + a_k Z_1^k\right]$$

since on $U_1 = \{Z_1 \neq 0\}$ the coordinate is $\tilde{z} = Z_0/Z_1$. Therefore the homogeneous polynomial corresponding to this section is $P = a_0 Z_0^k + a_1 Z_0^{k-1} Z_1 + \cdots + a_k Z_1^k$.

Example 1.13. In higher dimensional projective space, define $\mathcal{O}(k) \to \mathbb{P}^n$ by (U_i, t_{ij}) where $U_i = \{Z_i \neq 0\}$ and $t_{ij} : U_i \cap U_j \to \mathbb{C}^*$ is

$$t_{ij} = \left(\frac{Z_j}{Z_i}\right)^k$$

For example on $\mathcal{O}(1) \to \mathbb{P}^2$, with coordinates (U_0, z), $(U_1, \tilde{z})$ with $z = (Z_1/Z_0, Z_2/Z_0)$ and $\tilde{z} = (Z_0/Z_1, Z_2/Z_1)$, then $t_{10} = Z_0/Z_1 = \tilde{z}^1$.

1.2.2 Bundle constructions

We now describe some bundle constructions. Let $E \to M$ be a complex vector bundle of rank r with trivializations t_{UV}.

• Conjugate bundle. The complex vector bundle $\bar{E} \to M$ has trivializations $\overline{t_{UV}}$. Note that if $E \to M$ is a holomorphic vector bundle, then $\bar{E}$ is not a holomorphic bundle. However, the next constructions do produce holomorphic bundles if E is holomorphic.

• Dual bundle. Define $E^* \to M$ to be the bundle of rank r defined by trivializations $(t_{UV}^{-1})^T$. We use the following index notation: components of sections $s \in \Gamma(M, E)$ are denoted s^i, and components of sections $\varphi \in \Gamma(M, E^*)$ are denoted φ_i, so that the transformation laws reads

$$s^i_U = t_{UV}{}^i{}_k s^k_V, \quad \varphi^U_i = \varphi^V_k t_{VU}{}^k{}_i.$$

This is the dual bundle because sections $s \in \Gamma(M, E)$ and $\varphi \in \Gamma(M, E^*)$ can be paired together to form a function

$$\varphi(s) := (\varphi_i s^i) \in C^\infty(M, \mathbb{R}).$$

This is because the transformation laws imply

$$\varphi^U_i s^i_U = \varphi^V_i s^i_V$$

and so $(\varphi_i s^i)(p)$ is independent of the choice of trivialization. In matrix notation $Q = [t_{UV}]$, the transformation laws for $s \in \Gamma(E)$ and $\varphi \in \Gamma(E^*)$ are

$$s \mapsto Qs, \quad \varphi \mapsto [Q^{-1}]^T \varphi, \quad \varphi(s) = \varphi^T s.$$

In terms of local frames, if $\{e_i\}$ is a local frame for E, we denote the corresponding dual frame on E^* by $\{e^i\}$. This is defined as $e^i(e_j) = \delta^i{}_j$, and a section $\varphi \in \Gamma(E^*)$ is written as

$$\varphi = \varphi_i e^i.$$

The pairing $\varphi(s)$ can then be seen by the formula for the dual frame: $\varphi(s) = (\varphi_i e^i)(s^k e_k) = \varphi_j s^j$.

• Determinant bundle. The line bundle $\det E \to M$ is defined by the trivializations $\det t_{UV}$.

• Tensor product. If $E \to M$, $\tilde{E} \to M$ are vector bundles, then the bundle $E \otimes \tilde{E} \to M$ has trivializations $t_{UV} \otimes \tilde{t}_{UV}$. In components, if indices i, j denotes indices on E and indices α, β indices on $\tilde{E}$, then

$$s^{i\alpha}_U = t_{UV}{}^i{}_j \tilde{t}_{UV}{}^\alpha{}_\beta s_V{}^{j\beta}.$$

• Endomorphism bundle. We will later encounter sections of $E^* \otimes E = \operatorname{End} E^*$, and our convention for $h \in \Gamma(\operatorname{End} E^*)$ will be

$$h = h_\alpha{}^\beta \, e^\alpha \otimes e_\beta$$

so that the transformation law for components reads

$$[h_U]_\alpha{}^\beta = t_{VU}{}^\mu{}_\alpha [h_V]_\mu{}^\nu t_{UV}{}^\beta{}_\nu,$$

which in matrix notation $Q = [t_{UV}]$, for $u \in \Gamma(E)$ and $h \in \Gamma(\operatorname{End} E^*)$, is

$$u \mapsto Qu, \quad h \mapsto [Q^T]^{-1} h Q^T. \tag{1.7}$$

Note that h defines a map $h|_p : E^*|_p \to E^*|_p$ by $h_\alpha{}^\beta \varphi_\beta$. Verifying that this map is well-defined is a similar calculation as (1.5).

In fact, $h \in \Gamma(\operatorname{End} E^*)$ also defines an endomorphism of E by acting on the right as $u^T h$, or in index notation $u^\alpha h_\alpha{}^\beta$. That $u^T h$ transforms like a section follows from

$$(u^T h) \mapsto (Qu)^T ((Q^T)^{-1} h Q^T) = u^T h Q^T = Q(u^T h).$$

Thus $u^T h \in \Gamma(E)$. Thus, we will sometimes view $h \in \Gamma(\operatorname{End} E^*)$ with $h = h_\alpha{}^\beta$ as $h \in \Gamma(\operatorname{End} E)$.

• Divisor bundle. Let $Y \subset X$ be an analytic hypersurface. This means that near each $p \in Y$, there is neighborhood U such that $U \cap Y$ is locally given by the vanishing set of a holomorphic function. The theory of holomorphic functions (see e.g. [13]) implies that there exists the notion of a local defining function: this means that f is holomorphic with

$$U \cap Y = \{f = 0\}$$

and any other local holomorphic function g vanishing on Y factors as $g(z) = h(z) f(z)$ with h a local holomorphic function. The notion of local defining function is not unique: if f_1 and f_2 are local defining functions, then $f_1 = h f_2$ where h is a holomorphic function non-vanishing on U.

We can associate a line bundle $\mathcal{O}(Y) \to X$ in the following way. In a coordinate chart U, the submanifold Y appears as $Y \cap U = \{f_U(z) = 0\}$ where $f_U(z)$ is a local holomorphic function. The transition function of $\mathcal{O}(Y)$ is given by $t_{UV} = f_U / f_V$ on $U \cap V$. If $Y \cap U = \emptyset$, we can take $f_U = 1$.

• Note: if another choice of local defining function is taken, by (1.5) it follows that this defines an isomorphic bundle.

• Note: there is a global section $s \in H^0(\mathcal{O}(Y))$ given by the local data (U, s_U) with $s_U = f_U$, since $s_U = t_{UV} s_V$ is tautology.

Example 1.14. Let $P(Z_0, \ldots, Z_n)$ be a homogeneous polynomial of degree k, and let $Y = \{P = 0\} \subset \mathbb{P}^n$. Then $\mathcal{O}(Y) = \mathcal{O}(k)$. To see this, in the local chart $U_0 \subset \mathbb{P}^n$ the equation in coordinates (U_0, z) is $0 = P(Z_0, \ldots, Z_n)/Z_0^k = s_0$ and in the local chart U_1 the equation in coordinates $(U_1, \tilde{z})$ is $0 = P(Z_0, \ldots, Z_n)/Z_1^k = s_1$. The transition function t_{10} is then

$$t_{10} = \frac{P(Z_0, \ldots, Z_n)/Z_1^k}{P(Z_0, \ldots, Z_n)/Z_0^k} = \left[\frac{Z_0}{Z_1}\right]^k$$

which matches with the transition functions of $\mathcal{O}(k)$.

Let $Y \subseteq X$ be a smooth analytic hypersurface. This means that at $p \in U$, the local defining function $U \cap Y = \{f(z) = 0\}$ has the property that $\partial_i f(p) \neq 0$ for some coordinate direction ∂_i. In this case, there exists new holomorphic local coordinates $\{\tilde{z}^i\}$ such that (after possibly shrinking U)

$$U \cap Y = \{\tilde{z}^n = 0\}.$$

To see this, let $\{z^i\}$ be the original holomorphic coordinates and suppose after relabeling that $\frac{\partial f}{\partial z^n}(p) \neq 0$. By the holomorphic implicit function theorem, after possibly shrinking U we have

$$U \cap Y = \{(w^1, \ldots, w^{n-1}, \psi(w)) : w \in V\}$$

where $\psi : V \to \mathbb{C}$ is a holomorphic function and $V \subseteq \mathbb{C}^{n-1}$. New coordinates are then given by

$$\tilde{z}^1 = z^1, \ldots, \tilde{z}^{n-1} = z^{n-1}, \quad \tilde{z}^n = z^n - \psi(z),$$

and these satisfy $\tilde{z}^n(q) = 0$ if and only if $q \in V$.

• Canonical bundle. The canonical bundle of a complex manifold is $K_X = (\det T^{1,0}M)^*$. The transition functions on (U, z) an overlap $(\tilde{U}, \tilde{z})$ are $(\det \frac{\partial \tilde{z}}{\partial z})^{-1}$. Sections $\Omega \in \Gamma(K_X)$ are denoted

$$\Omega = f\, dz^1 \wedge \cdots \wedge dz^n,$$

and the transformation law is $\tilde{f} = \det(\frac{\partial z^p}{\partial \tilde{z}^q}) f$.

Proposition 1.15. *(Adjunction formula) Let $Y \subseteq X$ be a smooth analytic hypersurface.*

$$K_Y = (K_X \otimes \mathcal{O}(Y))|_Y$$

Proof. Locally Y is given by $\{z^n = 0\}$ for suitable holomorphic coordinates $(z^1, \ldots, z^n)$. On an overlap of open sets, suppose both $\tilde{z}^n = 0$ and $z^n = 0$ carve out Y. Then $\tilde{z}^n(z)$ is a holomorphic function of z which vanishes on $z^n = 0$, and so the theory of holomorphic functions implies that we can write

$$\tilde{z}^n(z) = z^n f(z)$$

for a holomorphic function f. For $i \leqslant n-1$, we compute

$$\frac{\partial \tilde{z}^n}{\partial z^i} = z^n \frac{\partial f}{\partial z^i}, \quad \frac{\partial \tilde{z}^n}{\partial z^n} = f + z^n \frac{\partial f}{\partial z^n}.$$

The transition function for $T^{1,0}X$ restricted to $Y = \{z^n = 0\}$ is then

$$\frac{\partial \tilde{z}}{\partial z}\bigg|_Y = \begin{bmatrix} A & * \\ 0 & f \end{bmatrix} \tag{1.8}$$

where A is transition function for $T^{1,0}Y$ using coords $z^1, \ldots, z^{n-1}$. Therefore

$$\det \frac{\partial \tilde{z}}{\partial z} = (\det A) f \tag{1.9}$$

and the transition functions give the bundle isomorphism

$$\det(T^{1,0}X)|_Y = (\det T^{1,0}Y) \otimes L,$$

where the line bundle L has transition function f. We can take the inverse to get the formula with $K_Y = (\det T^{1,0}Y)^{-1}$. Note that L has the same transition function as $\mathcal{O}(Y)$. By definition, if locally $z^n = 0$ and $\tilde{z}^n = 0$ carve out Y, the transition function of $\mathcal{O}(Y)$ is $\tilde{z}^n/z^n$ which in this case is $f(z)$. ☐

1.2.3 Constructions from the tangent bundle

We now list some bundle constructions which come from the holomorphic tangent bundle $T^{1,0}M$.

• Complexified tangent bundle. The conjugate of $T^{1,0}M$ is denoted $T^{0,1}M := \overline{T^{1,0}M}$. A local frame is given by

$$\left\{\frac{\partial}{\partial \bar{z}^1}, \dots, \frac{\partial}{\partial \bar{z}^n}\right\},$$

and sections denoted $V^{\bar{i}}\frac{\partial}{\partial \bar{z}^i}$. We can write the (x, y) coordinate basis in terms of the $(z, \bar{z})$ coordinate basis by the change of variables $z^k = x^k + iy^k$, $\bar{z}^k = x^k - iy^k$ and the chain rule:

$$\frac{\partial}{\partial x^k} = \frac{\partial}{\partial z^k} + \frac{\partial}{\partial \bar{z}^k}, \quad \frac{\partial}{\partial y^k} = i\left[\frac{\partial}{\partial z^k} - \frac{\partial}{\partial \bar{z}^k}\right].$$

It follows that the complexified tangent bundle $T_{\mathbb{C}}M$ can be written as a direct sum

$$T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M.$$

Example 1.16. We will sometimes use the notion of a complex structure $J : T_{\mathbb{C}}M \to T_{\mathbb{C}}M$. Given a complex manifold, the complex structure J is defined by setting

$$J|_{T^{1,0}M} = +i\mathrm{Id}, \quad J|_{T^{0,1}M} = -i\mathrm{Id},$$

or in other words, $J\frac{\partial}{\partial z^k} = i\frac{\partial}{\partial z^k}$ and $J\frac{\partial}{\partial \bar{z}^k} = -i\frac{\partial}{\partial \bar{z}^k}$. In components,

$$J^p{}_q = i\delta^p{}_q, \quad J^{\bar{p}}{}_{\bar{q}} = -i\delta^p{}_q, \quad J^p{}_{\bar{q}} = J^{\bar{p}}{}_q = 0,$$

in complex coordinates.

• Complexified cotangent bundle. We denote smooth sections of the dual of the holomorphic cotangent bundle $(T^{1,0}M)^*$ by $\Lambda^{1,0}(M)$, and holomorphic sections of this bundle by $H^0(M, (T^{1,0}M)^*)$. A local frame is given by

$$\{dz^1, \dots, dz^n\}$$

meaning $dz^k(\partial_{z^i}) = \delta^k{}_j$, and so that a section $\alpha \in \Lambda^{1,0}(M)$ is written $\alpha = \alpha_i dz^i$ and $\alpha(V) = \alpha_i V^i$ for $V = V^i\partial_{z^i} \in \Gamma(M, T^{1,0}M)$. Transformation laws for components and frames are

$$\tilde{\alpha}_i = \frac{\partial z^p}{\partial \tilde{z}^i}\alpha_p, \quad d\tilde{z}^k = \frac{\partial \tilde{z}^k}{\partial z^i}dz^i.$$

Denote $\Lambda^{0,1}(M) = \overline{\Lambda^{1,0}(M)}$. Complexified 1-forms $\Lambda^1_{\mathbb{C}}M$ can be decomposed as

$$\Lambda^1_{\mathbb{C}} = \Lambda^{1,0}(M) \oplus \Lambda^{0,1}(M),$$

from the decompositions (1.1)

$$dx^k = \frac{1}{2}(dz^k + d\bar{z}^k), \quad dy^k = \frac{1}{2i}(dz^k - d\bar{z}^k). \tag{1.10}$$

• Differential forms. Let $z^k = x^k + iy^k$ be local complex coordinates, and write $w = (x^1, \dots, x^n, y^1, \dots, y^n) \in \mathbb{R}^{2n}$. A differential form on M appears in w coordinates as

$$\eta = \frac{1}{k!}\eta_{i_1 \cdots i_k}\, dw^{i_1} \wedge \cdots \wedge dw^{i_k}.$$

From (1.10), we see that this can be written in the complex basis of dz, $d\bar{z}$. We will use the following convention for complex components:

$$\eta = \sum_{p+q=k} \eta^{p,q}$$

with

$$\eta^{p,q} = \frac{1}{p!q!}\eta_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q} dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}.$$

We call $\eta^{p,q}$ a (p,q)-form, denoted $\Lambda^{p,q}(M)$.

• Exterior derivative. The exterior derivative acting on a function is

$$df = \frac{\partial f}{\partial x^i}dx^i + \frac{\partial f}{\partial y^i}dy^i$$

which in complex coordinates becomes

$$df = \frac{\partial f}{\partial z^i}dz^i + \frac{\partial f}{\partial \bar{z}^i}d\bar{z}^i.$$

We write this as $df = \partial f + \bar{\partial} f$, with

$$\partial f = \frac{\partial f}{\partial z^i}dz^i, \quad \bar{\partial} f = \frac{\partial f}{\partial \bar{z}^i}d\bar{z}^i.$$

Similarly, the exterior derivative $d : \Lambda^k \to \Lambda^{k+1}$ on higher differential forms decomposes into types.

$$d = \partial + \bar{\partial}.$$

Acting on $\chi \in \Lambda^{p,q}(M)$, we have

$$\partial\chi = \frac{1}{p!q!}\frac{\partial}{\partial z^\ell}\chi_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q} dz^\ell \wedge dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}$$

and

$$\bar{\partial}\chi = \frac{1}{p!q!}\frac{\partial}{\partial \bar{z}^\ell}\chi_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q} d\bar{z}^\ell \wedge dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}$$

so that $\partial : \Lambda^{p,q} \to \Lambda^{p+1,q}$ and $\bar{\partial} : \Lambda^{p,q} \to \Lambda^{p,q+1}$.

Example 1.17. For $\alpha \in \Lambda^{1,1}$, we write $\alpha = \alpha_{j\bar{k}} dz^j \wedge d\bar{z}^k$ and

$$\begin{aligned} \partial\alpha &= \partial_\ell \alpha_{j\bar{k}} dz^\ell \wedge dz^j \wedge d\bar{z}^k \\ &= \frac{1}{2}(\partial_\ell \alpha_{j\bar{k}} - \partial_j \alpha_{\ell\bar{k}}) dz^\ell \wedge dz^j \wedge d\bar{z}^k \\ &= \frac{1}{2}(\partial\alpha)_{\ell j\bar{k}} dz^\ell \wedge dz^j \wedge d\bar{z}^k. \end{aligned} \tag{1.11}$$

and the components formula is

$$(\partial\alpha)_{\ell j\bar{k}} = \partial_\ell \alpha_{j\bar{k}} - \partial_j \alpha_{\ell\bar{k}}.$$

1.3 Geometry of bundles

1.3.1 Chern connection

Let $E \to M$ be a holomorphic vector bundle of rank r over a complex manifold. A hermitian metric on E is $H \in \Gamma(E^* \otimes \bar{E}^*)$ which is represented in a local frame $\{e_\alpha\}$ of E by

$$H = H_{\alpha\bar{\beta}}\, e^\alpha \otimes \overline{e^\beta}$$

with $H_{\alpha\bar{\beta}}(p)$ a positive-define $r \times r$ hermitian matrix at all points p. The hermitian condition is $\overline{H_{\alpha\bar{\beta}}} = H_{\beta\bar{\alpha}}$. Our conventions for the inner product on E given by H is

$$\langle u, v\rangle = u^i H_{i\bar{k}} \overline{v^k}, \quad u, v \in \Gamma(E)$$

so that $\langle u, \lambda v\rangle = \bar{\lambda}\langle u, v\rangle$. The hermitian condition is $\langle u, v\rangle = \overline{\langle v, u\rangle}$. The norm of a section is $|u|^2 = \langle u, u\rangle$.

We note that $\langle u, v\rangle$ does not depend on the choice of trivialization. Let $U, \tilde{U}$ be two trivializations of E with transition matrix $[Q] = Q^\alpha{}_\beta$ and denote components on $\tilde{U}$ with tildes, so for $u \in \Gamma(E)$ and $\varphi \in \Gamma(E^*)$ we have

$$\tilde{u}^\alpha = Q^\alpha{}_\beta u^\beta, \quad \tilde{\varphi}_\alpha = \varphi_\alpha (Q^{-1})^\alpha{}_\beta$$

and the transformation law on $H \in \Gamma(E^* \otimes \bar{E}^*)$ is

$$\tilde{H}_{\alpha\bar{\beta}} = (Q^{-1})^\mu{}_\alpha H_{\mu\bar{\nu}} \overline{(Q^{-1})^\nu{}_\beta}.$$

From here we can verify $\tilde{u}^\alpha \tilde{H}_{\alpha\bar{\beta}} \overline{\tilde{v}^\beta} = u^\alpha H_{\alpha\bar{\beta}} \overline{v^\beta}$. This can also be written using matrix notation:

$$\begin{aligned} \langle u, v\rangle &= u^T H \bar{v}, \\ \tilde{u} &= Qu, \quad \tilde{H} = (Q^{-1})^T H \overline{Q^{-1}}, \end{aligned} \tag{1.12}$$

and it is straightforward to verify that $\tilde{u}^T \tilde{H} \bar{\tilde{v}} = u^T H \bar{v}$. In other words, though the direction of u^α as a column vector is not a well-defined quantity (depends on the choice of trivialization), its norm $|u|_H$ is a measurable number.

The inverse of H is denoted in components as $H^{\bar{\alpha}\beta}$, so that $HH^{-1} = I$ becomes in components $H_{\alpha\bar{\beta}} H^{\bar{\beta}\gamma} = \delta_\alpha{}^\gamma$. The inverse H^{-1} produces a metric on $E^\star$.

$$\langle \psi, \varphi\rangle = H^{\bar{\alpha}\beta} \psi_\beta \overline{\varphi_\alpha}, \quad \psi, \varphi \in \Gamma(E^*).$$

Similarly as above, it can be verified that $\langle \psi_U, \varphi_U\rangle = \langle \psi_{\tilde{U}}, \varphi_{\tilde{U}}\rangle$, so that $\langle \psi, \varphi\rangle$ takes two sections and produces a global function on X.

The metric can be used to raise and lower indices. From $u^\alpha \in \Gamma(E)$, we will write

$$u_{\bar{\beta}} = u^\alpha H_{\alpha\bar{\beta}},$$

and $u_{\bar{\beta}}$ defines a section of $\bar{E}^*$. This is because if $u \mapsto Qu$ and $H \mapsto (Q^{-1})^T H \overline{Q^{-1}}$, then $u^T H \mapsto u^T H \overline{Q^{-1}} = (\overline{Q^{-1}})^T (u^T H)$. Said another way, given $u \in \Gamma(E)$ and a metric H, we obtain a dual element $u^* \in \Gamma(\bar{E}^*)$ defined by

$$u^*(\bar{v}) = \langle u, v\rangle_H.$$

Similarly, from $u_\alpha \in \Gamma(\bar{E}^*)$, then $u^\alpha = H^{\alpha\bar{\beta}} u_{\bar{\beta}}$ is a section of $\Gamma(E)$. We note that

$$u_\alpha v^\alpha = u^{\bar{\alpha}} v_{\bar{\alpha}}.$$

Definition 1.18. *The Chern connection of a metric H on a holomorphic bundle E is a map $\nabla : \Gamma(E) \to \Gamma(E \otimes (T_{\mathbb{C}} M)^*)$ given by*

$$\begin{aligned} \nabla_{\partial_k}(u^\alpha e_\alpha) &= (\partial_k u^\alpha + u^\beta (\partial_k H_{\beta\bar{\nu}}) H^{\bar{\nu}\alpha}) e_\alpha \\ \nabla_{\partial_{\bar{k}}}(u^\alpha e_\alpha) &= (\partial_{\bar{k}} u^\alpha) e_\alpha. \end{aligned} \tag{1.13}$$

We often just write this in components as

$$\begin{aligned} \nabla_{\bar{k}} u^\alpha &= \partial_{\bar{k}} u^\alpha, \\ \nabla_k u^\alpha &= \partial_k u^\alpha + u^\beta A_{k\beta}{}^\alpha, \quad A_{k\beta}{}^\alpha = \partial_k H_{\beta\bar{\nu}} H^{\bar{\nu}\alpha}, \end{aligned} \tag{1.14}$$

or without indices as

$$\nabla = (\partial + \partial H H^{-1}) + \bar{\partial}.$$

For $\nabla_{\bar{k}} s^\alpha$ to be a section, we need to verify that if $(\tilde{U}, \tilde{s}^\alpha)$ and (U, s^α) are two overlapping trivializations of E with $\tilde{s}^\alpha = Q^\alpha{}_\beta s^\beta$, then

$$\nabla_{\bar{k}} \tilde{s}^\alpha = Q^\alpha{}_\beta \nabla_{\bar{k}} s^\beta.$$

This is true because $\bar{\partial} Q^\alpha{}_\beta = 0$. It can also be checked directly that $\nabla_k s^\alpha$ is a section, namely

$$\nabla_k \tilde{s}^\alpha = Q^\alpha{}_\beta \nabla_k s^\beta,$$

by using the transformation law for H.

Recall that a general connection ∇ on a complex vector bundle E is a map $\nabla : \Gamma(E) \to \Gamma(E \otimes T^* M)$ such that $\nabla(a s_1 + b s_2) = a \nabla s_1 + b \nabla s_2$ and $\nabla(f s) = df \otimes s + f \nabla s$. We will also sometimes call ∇ a covariant derivative. The Chern connection is the most commonly used choice of connection on holomorphic bundles, and it is characterized by the following uniqueness statement:

Lemma 1.19. *Let (E, H) be a holomorphic bundle with hermitian metric. The Chern connection is the unique connection satisfying $\nabla^{0,1} = \bar{\partial}$ and*

$$\partial_k \langle u, v \rangle = \langle \nabla_k u, v \rangle + \langle u, \nabla_{\bar{k}} v \rangle. \tag{1.15}$$

Proof. Let ∇ be a connection satisfying (1.15) with $\nabla^{0,1} = \bar{\partial}$. We will solve for $\nabla^{1,0}$. Our notation for the unknown connection is

$$\nabla_{\partial_k} e_\alpha = A_{k\alpha}{}^\beta e_\beta$$

where A are unknown coefficients to be solved. In other words

$$\nabla_k u^\alpha = \partial_k u^\alpha + A_{k\beta}{}^\alpha u^\beta.$$

If we require (1.15), then in coordinates this becomes

$$\partial_k (H_{i\bar{j}} u^i \overline{v^j}) = H_{i\bar{j}} \nabla_k u^i \overline{v}^{\bar{j}} + H_{i\bar{j}} u^i \overline{\partial_{\bar{k}} v^j}$$

which simplies to

$$\partial_k H_{i\bar{j}} u^i \overline{v^j} = H_{r\bar{j}} A_{ki}{}^r u^i \overline{v^j}$$

If this is true for all sections u, v, then

$$\partial_k H_{i\bar{j}} = A_{ki}{}^r H_{r\bar{j}}$$

and solving for A gives

$$A_{ki}{}^\ell = \partial_k H_{i\bar{j}} H^{\bar{j}\ell}$$

or $A = \partial H H^{-1}$. ☐

There is a formula for how the Chern connection changes when changing the metric. Let $\hat{H}$ and H be two metrics on E. Let $A = \partial H H^{-1}$ and $h = H\hat{H}^{-1}$. Then

$$\begin{aligned} A &= \partial H H^{-1} \\ &= \partial(h\hat{H})\hat{H}^{-1}h^{-1} \\ &= \partial h h^{-1} + h\hat{A}h^{-1} \\ &= \partial h h^{-1} + h\hat{A}h^{-1} + (\hat{A} - \hat{A}hh^{-1}) \\ &= \hat{A} + \hat{\nabla} h h^{-1} \end{aligned} \tag{1.16}$$

where $\hat{\nabla} h = \partial h + h\hat{A} - \hat{A}h$ in matrix notation, or using index notation for the component of $h_\alpha{}^\beta = H_{\alpha\bar{\mu}}\hat{H}^{\bar{\mu}\beta}$, then

$$\nabla_i h_\alpha{}^\beta = \partial_i h_\alpha{}^\beta + h_\alpha{}^\gamma A_{i\gamma}{}^\beta - A_{i\alpha}{}^\gamma h_\gamma{}^\beta. \tag{1.17}$$

Here is the reason for the term with a minus sign. Let ∇ be a connection on E acting on sections $u \in \Gamma(E)$ by

$$\nabla_i u^\alpha = \partial_i u^\alpha + u^\beta A_{i\beta}{}^\alpha, \quad A_{i\beta}{}^\alpha = \partial_i H_{\beta\bar{\nu}} H^{\bar{\nu}\alpha}.$$

The induced dual connection acting on $\varphi \in \Gamma(E^*)$ is defined with a minus sign:

$$\nabla_i \varphi_\alpha = \partial_i \varphi_\alpha - A_{i\alpha}{}^\beta \varphi_\beta.$$

This minus sign is introduced so that we can differentiate contracted indices using the product rule

$$\partial_i(u^\alpha \varphi_\alpha) = (\nabla_i u^\alpha)\varphi_\alpha + u^\alpha(\nabla_i \varphi_\alpha).$$

The formula (1.17) follows from the rule for covariant differentiation where each upper index receives a $+A$ term and each lower index receives a $-A$ term. As another example,

$$\nabla_i T^\alpha{}_{\beta\gamma} = \partial_i T^\alpha{}_{\beta\gamma} + T^\mu{}_{\beta\gamma} A_{i\mu}{}^\alpha - T^\alpha{}_{\mu\gamma} A_{i\beta}{}^\mu - T^\alpha{}_{\beta\mu} A_{i\gamma}{}^\mu,$$

for $T \in \Gamma(E \otimes E^* \otimes E^*)$. Sections of $\bar{E}$ follow the rules

$$\nabla_i u^{\bar{\alpha}} = \partial_i u^{\bar{\alpha}}, \quad \nabla_{\bar{i}} u^{\bar{\alpha}} = \partial_{\bar{i}} u^{\bar{\alpha}} + u^{\bar{\nu}} \overline{A_{i\nu}{}^\alpha}$$

so that $\nabla \bar{u} = \overline{\bar{\nabla} u}$ for $u \in \Gamma(E)$. For example,

$$\nabla_i G_{\alpha\bar{\beta}} = \partial_i G_{\alpha\bar{\beta}} - A_{i\alpha}{}^\nu G_{\nu\bar{\beta}}$$

for $G \in \Gamma(E^* \otimes \bar{E}^*)$. From this formula and $A_i = (\partial_i H)H^{-1}$, we see that

$$\nabla_i H_{\alpha\bar{\beta}} = 0, \quad \nabla_{\bar{i}} H_{\alpha\bar{\beta}} = 0$$

when ∇ is the Chern connection of H.

1.3.2 Curvature

The curvature of the Chern connection defines a notion of Hessian of a metric tensor H. The local combinations $\partial_j \partial_{\bar{k}} H_{\alpha\bar{\beta}}$ do not transform as the section of any bundle, and taking covariant derivatives gives zero: $\nabla_i H_{\alpha\bar{\beta}} = 0$. The curvature tensor is a way to encode second derivatives of the metric.

Definition 1.20. *Let (E, H) be a holomorphic vector bundle with metric. The curvature of the Chern connection $F \in \Gamma(\Lambda^{1,1} \otimes \mathrm{End}\, E)$ is given by $F = \bar{\partial}(\partial H H^{-1})$. In components*

$$F = F_\beta{}^\alpha{}_{j\bar{k}}\, e_\alpha \otimes e^\beta\, dz^j \wedge d\bar{z}^k,$$

the definition is

$$F_\alpha{}^\beta{}_{j\bar{k}} = -\partial_{\bar{k}}(\partial_j H_{\alpha\bar{\mu}} H^{\bar{\mu}\beta}),$$

or

$$F_{j\bar{k}} = -\partial_{\bar{k}}(\partial_j H H^{-1}),$$

without showing the endomorphism indices.

The action of $F_{j\bar{k}} \in \Gamma(\mathrm{End}\, E)$ on $u \in \Gamma(E)$ is

$$u^\alpha \mapsto u^\beta F_\beta{}^\alpha{}_{j\bar{k}}.$$

We now verify that the formula for $F_{j\bar{k}}$ gives a well-defined section of $\mathrm{End}\, E$. The transformation law $\tilde{H} = (Q^{-1})^T H \overline{Q^{-1}}$ implies

$$\begin{aligned}
\tilde{F}_{j\bar{k}} &= -\partial_{\bar{k}}\Big[\partial_j((Q^{-1})^T H \overline{Q^{-1}})(\overline{Q} H^{-1} Q^T)\Big] \\
&= -\partial_{\bar{k}}\Big[(-Q^T \partial_j Q^T)\Big] + -\partial_{\bar{k}}\Big[((Q^{-1})^T \partial_j H H^{-1} Q^T)\Big] \\
&= (Q^{-1})^T\Big[-\partial_{\bar{k}}(\partial_j H H^{-1})\Big] Q^T
\end{aligned} \tag{1.18}$$

using $\bar{\partial} Q = 0$, $\partial \bar{Q} = 0$. This matches with (1.7), and so $F_{j\bar{k}} \in \Gamma(\mathrm{End}\, E)$ acting on sections of E on the right.

Remark 1.21. Let $L \to M$ be a holomorphic line bundle. Then H is a 1×1 matrix, and the transformation law for a metric reads

$$\tilde{H} = \frac{1}{|t_{\tilde{U}U}|^2} H. \tag{1.19}$$

The formula for the curvature is

$$F_{j\bar{k}} = -\partial_{\bar{k}} \partial_j \log H, \quad iF = -i\partial\bar{\partial} \log H \in \Lambda^{1,1}(M),$$

and it can also be checked directly that $\tilde{F}_{j\bar{k}} = F_{j\bar{k}}$.

Remark 1.22. The formula for change of curvature is

$$F = \hat{F} + \bar{\partial}(\hat{\nabla} h h^{-1}).$$

where $H, \hat{H}$ are two metrics and $h = H\hat{H}^{-1}$. This is because (1.16) implies

$$\bar{\partial} A = \bar{\partial}\hat{A} + \bar{\partial}(\hat{\nabla} h h^{-1}).$$

Remark 1.23. The formula for Chern curvature is consistent with the general formula for the curvature of a connection. In general, the curvature of a connection ∇ on E is $F = dA - A \wedge A$, $F = \frac{1}{2} F_j{}^i{}_{\mu\nu} dx^\mu \wedge dx^\nu \otimes e^j \otimes e_i$, with

$$F_j{}^i{}_{\mu\nu} = \partial_\mu A_{\nu j}{}^i - \partial_\nu A_{\mu j}{}^i - A_{\mu j}{}^r A_{\nu r}{}^i + A_{\mu j}{}^r A_{\mu r}{}^i.$$

For the Chern connection, $\nabla^{0,1} = \bar{\partial}$, so $A_{\bar{i}\alpha}{}^\beta = 0$. Therefore $F_\alpha{}^\beta{}_{\bar{k}\bar{j}} = 0$, and since the Chern connection is unitary then $F_\alpha{}^\beta{}_{kj} = 0$ which can also be checked directly. Therefore F only has mixed $(1,1)$ form indices, and

$$F_j{}^i{}_{m\bar{n}} = -\partial_{\bar{n}} A_{mj}{}^i$$

since mixed connection terms are zero.

Example 1.24. Fubini-Study metric on $\mathcal{O}(1) \to \mathbb{P}^1$. Recall that $\mathbb{P}^1$ is covered by two trivializations (U_0, z), $(U_1, \tilde{z})$, with change of coordinates $\tilde{z} = z^{-1}$ and sections $s \in \Gamma(\mathcal{O}(1))$ transform as $\tilde{s} = \tilde{z}s$. The Fubini-Study metric is defined as

$$h(z) = (1 + |z|^2)^{-1}, \quad \tilde{h}(\tilde{z}) = (1 + |\tilde{z}|^2)^{-1}.$$

This transforms correctly as (1.19): $\tilde{h} = (1/|\tilde{z}|^2)h$. In other words, the norm

$$|s|_h^2 = s\bar{s}h$$

gives the same result in either trivialization. The curvature is a 2-form $iF \in \Lambda^{1,1}$ with components

$$F_{z\bar{z}} = -\partial_{\bar{z}}\partial_z \log h.$$

We compute

$$F_{z\bar{z}} = \partial_{\bar{k}} \frac{\bar{z}}{1+|z|^2} = \frac{1}{1+|z|^2} - \frac{|z|^2}{(1+|z|^2)^2} = (1+|z|^2)^{-2} > 0.$$

Therefore iF is a closed positive $(1,1)$ form; this is a Kähler metric on $\mathbb{P}^1$. The Fubini-Study Kähler metric is sometimes denoted

$$\omega_{FS} = i\partial\bar{\partial}\log(1+|z|^2).$$

1.3.3 Hermitian geometry

In this section, we focus on metrics and curvature on the holomorphic tangent bundle $T^{1,0}X$. We will denote a hermitian metric on $T^{1,0}M$ by g, with components $g_{j\bar{k}}$.

$$g = g_{j\bar{k}}\, dz^j \otimes d\bar{z}^k.$$

The collection of local matrices $(U, g_{j\bar{k}})$ are related by

$$\tilde{g}_{j\bar{k}} = \frac{\partial z^\ell}{\partial \tilde{z}^j} g_{\ell\bar{m}} \overline{\frac{\partial z^m}{\partial \tilde{z}^k}}.$$

Locally we view g as a matrix, e.g. in 2 dimensions

$$g = \begin{bmatrix} g_{1\bar{1}} & g_{1\bar{2}} \\ g_{2\bar{1}} & g_{2\bar{2}} \end{bmatrix}.$$

The inner product on $T^{1,0}M$ is then

$$g(V,W) = V^j g_{j\bar{k}} \bar{W}^{\bar{k}}, \quad V = V^i \frac{\partial}{\partial z^i}, \ W = W^i \frac{\partial}{\partial z^i}.$$

A Kähler metric on M is a metric satisfying $\partial_i g_{j\bar{k}} = \partial_j g_{i\bar{k}}$. Another way to express this is to associate a $(1,1)$ form

$$\omega = i g_{j\bar{k}} dz^j \wedge d\bar{z}^k$$

and require $d\omega = 0$. Note that the factor of i is included in the definition so that ω is real: $\bar{\omega} = \omega$. Direct calculation gives

$$\frac{\omega^n}{n!} = \det g_{j\bar{k}} \, idz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge idz^n \wedge d\bar{z}^n, \tag{1.20}$$

and $\omega^n/n!$ is a nowhere vanishing top form which will later be used for integration.

Remark 1.25. A hermitian metric $g = g_{j\bar{k}}$ on $T^{1,0}M$ produces a Riemannian metric on the real tangent bundle TM, which we also denote by g. Let us temporarily write this metric as $g_{\mathbb{R}}$. Let z^j be holomorphic coordinates (indices j,k) and $w^\alpha = (z^1, \dots, z^n, \bar{z}^1, \dots, \bar{z}^n)$ be full coordinates (indices α, β). Then we define

$$g_{\mathbb{R}}(\partial_\alpha, \partial_\beta) = g_{\alpha\beta}$$

where we declare

$$g_{jk} = g_{\bar{j}\bar{k}} = 0, \quad g_{j\bar{k}} = g_{\bar{k}j}.$$

In other words,

$$g_{\mathbb{R}}(X^\alpha \partial_\alpha, Y^\beta \partial_\beta) = g_{j\bar{k}} X^j Y^{\bar{k}} + g_{\bar{j}k} X^{\bar{j}} Y^k$$

where $X = X^i \frac{\partial}{\partial z^i} + X^{\bar{i}} \frac{\partial}{\partial \bar{z}^i}$. From this perspective, the $(1,1)$-form ω may be defined as $\omega(X,Y) = g_{\mathbb{R}}(JX, Y)$ since $\omega(\partial_j, \partial_{\bar{k}}) = i g_{j\bar{k}}$.

We can also write the metric g (we now stop using the notation $g_{\mathbb{R}}$) in terms of $q = (x^1, y^1, \dots, x^n, y^n)$ coordinates where $z^k = x^k + iy^k$. We give the details in complex dimension 1. Let $z = x + iy$ be complex coordinates, and we would like to write

$$g = \begin{bmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{bmatrix}$$

A hermitian metric is given by a mixed type $g_{z\bar{z}} > 0$, and we declare $g_{zz} = 0$ and $g_{\bar{z}\bar{z}} = 0$. Changing coordinates

$$\begin{aligned} g_{xx} &= g(\partial_z + \partial_{\bar{z}}, \partial_z + \partial_{\bar{z}}) = 2g_{z\bar{z}}, \\ g_{xy} &= g(\partial_z + \partial_{\bar{z}}, i(\partial_z - \partial_{\bar{z}})) = 0, \\ g_{yy} &= g(i(\partial_z - \partial_{\bar{z}}), i(\partial_z - \partial_{\bar{z}})) = 2g_{z\bar{z}}. \end{aligned} \tag{1.21}$$

The corresponding Riemannian metric in (x, y) coordinates is then diagonal with

$$g = \begin{bmatrix} 2g_{z\bar{z}} & 0 \\ 0 & 2g_{z\bar{z}} \end{bmatrix}.$$

Note: there is a way to go from a Riemannian metric on TM to a hermitian metric on $T^{1,0}M$, but this requires that g is compatible with the complex structure so that $g(JX, JX) = g(X, X)$ and we omit the details.

We will use a normalization factor of $p!q!$ for the inner product defined on (p, q) forms. For $\varphi, \psi \in \Lambda^{p,q}$ with

$$\varphi = \frac{1}{p!q!} = \varphi_{\alpha_1 \cdots \alpha_p \bar{\beta}_1 \cdots \bar{\beta}_q} \, dz^{\alpha_1} \wedge \cdots \wedge dz^{\alpha_p} \wedge d\bar{z}^{\beta_1} \wedge \cdots \wedge d\bar{z}^{\beta_q},$$

and

$$\bar{\varphi}^{\alpha_1 \cdots \alpha_p \bar{\beta}_1 \ldots \bar{\beta}_q} = g^{\bar{a}_1 \alpha_1} \cdots g^{\bar{a}_p \alpha_p} g^{\bar{\beta}_1 b_1} \cdots g^{\bar{\beta}_q b_q} \overline{\varphi_{a_1 \cdots a_p \bar{b}_1 \cdots \bar{b}_q}}$$

then we will use the convention

$$\langle \varphi, \psi \rangle = \frac{1}{p!q!} \varphi_{\alpha_1 \cdots \alpha_p \bar{\beta}_1 \cdots \bar{\beta}_q} \bar{\psi}^{\alpha_1 \cdots \alpha_p \bar{\beta}_1 \ldots \bar{\beta}_q}. \tag{1.22}$$

The Chern connection ∇ on the bundle $T^{1,0}X \to X$ is defined by

$$\nabla_{\bar{k}} V^i = \partial_{\bar{k}} V^i, \quad \nabla_k V^i = \partial_k V^i + V^p \Gamma_{kp}{}^i, \quad \Gamma_{kp}{}^i = \partial_k g_{p\bar{\ell}} g^{\bar{\ell} i}.$$

and a direct check shows that $\nabla g = 0$ and $\nabla J = 0$.

The curvature of the Chern connection on the tangent bundle will be denoted $R \in \Lambda^{1,1}(\mathrm{End}\, T^{1,0}M)$, so that

$$R_j{}^i{}_{m\bar{n}} = -\partial_{\bar{n}} \Gamma_{mj}{}^i$$

since $A_{mj}{}^i = \Gamma_{mj}{}^i$. Explicitly in terms of the metric, the curvature of the Chern connection is given by

$$R_j{}^i{}_{m\bar{k}} = -\partial_{\bar{k}} (\partial_m g_{j\bar{p}} g^{\bar{p} i}).$$

We can lower the second index

$$R_{j\bar{p}m\bar{k}} = R_j{}^i{}_{m\bar{k}} g_{i\bar{p}}$$

and the explicit formula in terms of the metric is

$$R_{j\bar{p}m\bar{k}} = -\partial_{\bar{k}} \partial_m g_{j\bar{p}} + (\partial_m g_{j\bar{q}}) g^{\bar{q} a} (\partial_{\bar{k}} g_{a\bar{p}})$$

using the variation formula for the inverse of a matrix $\delta A^{-1} = -A^{-1}(\delta A)A^{-1}$. The conjugate of this is

$$\overline{R_{j\bar{p}m\bar{k}}} = -\partial_{\bar{m}}\partial_k g_{p\bar{j}} + (\partial_k g_{p\bar{a}})g^{\bar{a}q}(\partial_{\bar{m}} g_{q\bar{j}})$$

so

$$\overline{R_{j\bar{p}m\bar{k}}} = R_{p\bar{j}k\bar{m}}, \quad \overline{R_j{}^i{}_{m\bar{k}}} = g^{\bar{i}p}R_{p\bar{j}k\bar{m}} = R^{\bar{i}}{}_{\bar{j}k\bar{m}}.$$

The curvature appears when exchanging covariant derivatives. We will write

$$[\nabla_m, \nabla_{\bar{k}}]V^i = \nabla_m\nabla_{\bar{k}}V^i - \nabla_{\bar{k}}\nabla_m V^i$$

where $\nabla_m\nabla_{\bar{k}}V^i$ means: let $W_{\bar{k}}{}^i = \nabla_{\bar{k}}V^i$ be a tensor and compute the components $\nabla_m W_{\bar{k}}{}^i$. The commutator formula is

$$[\nabla_m, \nabla_{\bar{k}}]V^i = V^p R_p{}^i{}_{m\bar{k}}. \tag{1.23}$$

Here is the check:

$$\begin{aligned} [\nabla_m, \nabla_{\bar{k}}]V^i &= \partial_m\nabla_{\bar{k}}V^i + \Gamma_{m\ell}{}^i\nabla_{\bar{k}}V^\ell - \partial_{\bar{k}}\nabla_m V^i \\ &= \partial_m\partial_{\bar{k}}V^i + \Gamma_{m\ell}{}^i\partial_{\bar{k}}V^\ell - \partial_{\bar{k}}(\partial_m V^i + \Gamma_{m\ell}{}^i V^\ell) \\ &= -\partial_{\bar{k}}\Gamma_{m\ell}{}^i V^\ell = R_\ell{}^i{}_{m\bar{k}}V^\ell. \end{aligned} \tag{1.24}$$

Remark 1.26. A similar calculation gives that the commutator $[\nabla_m, \nabla_k]V^i$ involves another tensor, the torsion tensor, but we will not need this formula.

Taking the conjugate of (1.23) gives

$$[\nabla_k, \nabla_{\bar{m}}]V^{\bar{i}} = -V^{\bar{p}}R^{\bar{i}}{}_{\bar{p}k\bar{m}}.$$

We can lower the index by introducing $g_{a\bar{i}}$, since $\nabla_k g_{a\bar{i}} = 0$.

$$[\nabla_k, \nabla_{\bar{m}}]V_a = -R_a{}^p{}_{k\bar{m}}V_p.$$

Similarly

$$[\nabla_k, \nabla_{\bar{m}}]V_{\bar{a}} = R^{\bar{p}}{}_{\bar{a}m\bar{k}}V_{\bar{p}}.$$

Let

$$R_{j\bar{k}} = R_p{}^p{}_{j\bar{k}} = g^{\bar{q}p}R_{p\bar{q}j\bar{k}}$$

and define the Chern-Ricci form by

$$i\mathrm{Ric}_\omega = iR_{j\bar{k}}\, dz^j \wedge d\bar{z}^k.$$

Viewing $R \in \Lambda^{1,1}(\mathrm{End}\, T^{1,0}X)$, we have $i\mathrm{Ric}_\omega = i\mathrm{Tr}\, R \in \Lambda^{1,1}(X)$ where we trace out the endomorphism indices and retain the 2-form indices. Using the general formula for the derivative of the determinant of an invertible hermitian matrix A,

$$\left.\frac{d}{dt}\right|_{t=0} \det A(t) = \det A(0)\, \mathrm{Tr}\, [A(0)^{-1}\dot{A}(0)]$$

we obtain $\partial_j \log\det g = g^{\bar{r}p}\partial_j g_{p\bar{r}}$, and so

$$R_{j\bar{k}} = -\partial_{\bar{k}}\partial_j \log\det g, \tag{1.25}$$

and

$$i\mathrm{Ric}_\omega = -i\partial\bar\partial \log\det g.$$

This expression can also be connected to Riemannian geometry: in the case when $g_{i\bar j}$ is a Kähler metric, then it turns out that $R_{j\bar k}$ is the Levi-Civita Ricci tensor of the Riemannian metric g on TX. This calculation can be found in Kähler's original paper (see p.178 in [17], where (1.25) is described as "very elegant"), and it is one of the main motivations for the field of Kähler geometry.

We note that when g is a general hermitian metric on $T^{1,0}X$, the Chern-Ricci curvature $R_{j\bar k}$ (1.25) is different than the Riemannian Levi-Civita Ricci tensor.

Remark 1.27. From the point of view of Riemannian geometry, it looks like $R_{j\bar k} = R_p{}^p{}_{j\bar k}$ is tracing the wrong index in the definition of the Ricci curvature and should be zero. This is not the case because we are only tracing over holomorphic indices. Tracing over all real indices does indeed give zero. Let α, β represents real coordinates $x^\alpha = (z^1, \dots, z^n, \bar z^1, \dots, \bar z^n)$ and a, b, j, k represents holomorphic coordinates z^i, and let $R_{\alpha\beta\gamma\mu}$ be the Riemannian curvature tensor. Then

$$g^{\alpha\beta} R_{\alpha\beta j\bar k} = g^{a\bar b} R_{a\bar b j\bar k} + g^{\bar a b} R_{\bar a b j\bar k} = g^{a\bar b}(R_{a\bar b j\bar k} + R_{\bar b a j\bar k}) = 0,$$

since the Riemannian curvature tensor satisfies $R_{\alpha\beta\gamma\mu} = -R_{\beta\alpha\gamma\mu}$. But the trace

$$g^{a\bar b} R_{a\bar b j\bar k}$$

is not summing over all coordinate indices α, β, and need not be zero.

Example 1.28. We start with the line bundle $\mathcal{O}(1) \to \mathbb{P}^n$ with trivializations $U_i = \{Z_i \neq 0\}$. Recall that the transition functions $(U_i \cap U_j, t_{ij})$ are $t_{ij} = Z_j/Z_i$.

• Show that the collection (U_i, h_i) with

$$h_i = \frac{|Z_i|^2}{\sum_k |Z_k|^2}$$

defines a metric on $\mathcal{O}(1) \to \mathbb{P}^n$. We call h the Fubini-Study metric.

• Compute the curvature $iF = i\partial\bar\partial \log h$ of the Fubini-Study metric, and show that in local coordinates on U_i then

$$iF = i\partial\bar\partial \log(1 + |z|^2), \quad |z|^2 = \sum_{i=1}^n |z^i|^2. \tag{1.26}$$

We will show below that $iF := \omega_{FS}$ is a Kähler metric on the base $\mathbb{P}^n$.

• We also sometimes refer to the following expression

$$\omega_{FS} = i\partial\bar\partial \log(1 + |z|^2), \quad \omega_{FS} = i g_{j\bar k} dz^j \wedge d\bar z^k,$$

as the Fubini-Study metric on $\mathbb{P}^n$. We see that $d\omega_{FS} = 0$, and to show ω_{FS} is a Kähler metric, we need to verify that $g_{j\bar k}$ is a positive-definite matrix. A computation gives

$$g_{j\bar k} = \frac{(1 + |z|^2)\delta_{jk} - z_k \bar z_j}{(1 + |z|^2)^2}.$$

For $\xi \in \mathbb{C}^n$, we have

$$\begin{aligned} g_{j\bar{k}}\xi^j\bar{\xi}^k &= \frac{(1+|z|^2)|\xi|^2 - |z_k\bar{\xi}^k|^2}{(1+|z|^2)^2} \\ &\geqslant \frac{(1+|z|^2)|\xi|^2 - |z|^2|\xi|^2}{(1+|z|^2)^2} \\ &= \frac{|\xi|^2}{(1+|z|^2)^2} > 0. \end{aligned} \tag{1.27}$$

Therefore g is a hermitian metric.

• Compute the Chern-Ricci curvature of ω_{FS}. A computation of the determinant gives

$$\det g_{j\bar{k}} = (1+|z|^2)^{-(n+1)}.$$

From here, one can compute that

$$R_{j\bar{k}} = (n+1)g_{j\bar{k}}.$$

Another way to write this is $i\mathrm{Ric}(\omega_{FS}) = (n+1)\omega_{FS}$. Kähler metrics satisfying $i\mathrm{Ric}(\omega) = \lambda\omega$ for $\lambda \in \mathbb{R}$ are said to be Kähler-Einstein.

Example 1.29. Suppose X is a complex manifold with an embedding into projective space $i : X \to \mathbb{P}^N$. Then $i^*\omega_{FS}$ is Kähler metric on X. In other words, projective manifolds are Kähler. For an interesting converse, see the Kodaira embedding theorem: this states that a compact Kähler manifold (X,ω) with $[\omega] \in H^2(X,\mathbb{Z})$ admits an embedding $i : X \to \mathbb{P}^N$ into some projective space $\mathbb{P}^N$.

2 Kähler Manifolds

This section will follow the textbook by Kodaira-Morrow [20].

2.1 Hodge theory

We start with the definition of the Dolbeault cohomology groups. Let X be a compact complex manifold and $E \to X$ a holomorphic vector bundle. Let $\Lambda^{p,q}(E)$ be smooth (p,q) forms with coefficients in E, so that sections $s \in \Lambda^{p,q}(E)$ are of the form $s = u \otimes \eta$ with $u \in \Gamma(E)$ and $\eta \in \Lambda^{p,q}(X)$. The Dolbeault cohomology groups are defined

$$H^q(X,E) = \frac{\ker(\bar\partial : \Lambda^{0,q}(E) \to \Lambda^{0,q+1}(E))}{\mathrm{im}\,(\bar\partial : \Lambda^{0,q-1}(E) \to \Lambda^{0,q}(E))}$$

Letting $E = \Omega^p$ be the holomorphic bundle of $(p,0)$ forms, we also define

$$H^{p,q}_{\bar\partial}(X) := H^q(X,\Omega^p) = \frac{\ker(\bar\partial : \Lambda^{p,q}(X) \to \Lambda^{p,q+1}(X))}{\mathrm{im}\,(\bar\partial : \Lambda^{p,q-1}(X) \to \Lambda^{p,q}(X))}$$

and the Hodge numbers are

$$h^{p,q} = \dim H^{p,q}_{\bar\partial}(X).$$

The theme of Hodge theory is to represent the cohomology class $[\varphi]$ by a unique optimal representative $\varphi_0 \in [\varphi]$. The selected representative from the equivalence class is found by solving an elliptic PDE. In this particular case, we will look for solutions $\varphi_0 \in [\varphi]$ to the Laplace equation $\Delta_{\bar\partial}\varphi_0 = 0$.

To define the Laplacian, we must equip (E,X) with metrics: let H be a metric on E and g a metric on the base X. As described in earlier sections, this defines a pointwise inner product on $\varphi_1, \varphi_2 \in \Lambda^{p,q}(E)$:

$$\langle \varphi_1, \varphi_2 \rangle_{g,H} : X \to \mathbb{C}.$$

The L^2 inner product on $\Lambda^{p,q}(E)$ is then

$$(\varphi_1,\varphi_2)_{L^2} = \int_X \langle \varphi_1, \varphi_2 \rangle_{g,H} \frac{\omega^n}{n!}.$$

The Dolbeault operator $\bar\partial : \Lambda^{p,q}(E) \to \Lambda^{p,q+1}(E)$ has an L^2-adjoint denoted $\bar\partial^\dagger : \Lambda^{p,q}(E) \to \Lambda^{p,q-1}(E)$. The adjoint satisfies

$$(\bar\partial\varphi_1,\varphi_2)_{L^2} = (\varphi_1, \bar\partial^\dagger\varphi_2)_{L^2}, \quad \varphi_1 \in \Lambda^{p,q-1}(E),\ \varphi_2 \in \Lambda^{p,q}(E).$$

The $\bar\partial$-Laplacian $\Delta_{\bar\partial} : \Lambda^{p,q}(E) \to \Lambda^{p,q}(E)$ is then defined by

$$\Delta_{\bar\partial} = \bar\partial\bar\partial^\dagger + \bar\partial^\dagger\bar\partial.$$

There is an explicit formula for the adjoint in local coordinates. Roughly speaking, $\bar\partial$ is a like a curl operator while $\bar\partial^\dagger$ is like a divergence operator. We give the formula without the additional vector bundle $E \to X$, so that we consider usual (p,q) forms with adjoint $\bar\partial^\dagger : \Lambda^{p,q}(X) \to \Lambda^{p,q-1}(X)$. The formula in local coordinates is

$$(\bar\partial^\dagger\varphi)^{\bar I J} = -(\det g)^{-1}\partial_p\bigg[(\det g)\varphi^{p\bar I J}\bigg], \quad \varphi \in \Lambda^{p,q}(X). \tag{2.1}$$

Here $I = i_1 \cdots i_p$, $J = j_1 \cdots j_q$ are multi-indices, and we raise indices using the metric: e.g. if $\varphi = \frac{1}{q!}\varphi_{\bar{J}} d\bar{z}^J$ is $(0,q)$ form, then $\varphi^J = g^{j_1\bar{a}_1} \cdots g^{j_q\bar{a}_q}\varphi_{\bar{a}_1\cdots\bar{a}_q}$, or e.g. if $\varphi = \varphi_{r\bar{s}} dz^r \wedge d\bar{z}^s$ is a $(1,1)$-form, then $\varphi^{\bar{i}j} = g^{\bar{i}r} g^{\bar{s}j}\varphi_{r\bar{s}}$.

Let us verify formula (2.1) in the special case of $(0,2)$ forms. Let $\theta \in \Lambda^{0,2}(X)$ and $\varphi \in \Lambda^{0,1}(X)$. Then

$$\varphi = \varphi_{\bar{i}} d\bar{z}^i, \quad (\bar{\partial}\varphi)_{\bar{j}\bar{k}} = \partial_{\bar{j}}\varphi_{\bar{k}} - \partial_{\bar{k}}\varphi_{\bar{j}}.$$

Using the inner product (1.22) we have

$$\langle \bar{\partial}\varphi, \theta \rangle_g = \frac{1}{2} g^{\bar{j}i} g^{\bar{k}\ell} (\partial_{\bar{j}}\varphi_{\bar{k}} - \partial_{\bar{k}}\varphi_{\bar{j}}) \overline{\theta_{\bar{i}\bar{\ell}}} = \partial_{\bar{j}}\varphi_{\bar{k}} \bar{\theta}^{\bar{j}\bar{k}}$$

and

$$\partial_{\bar{j}}\varphi_{\bar{k}} \bar{\theta}^{\bar{j}\bar{k}} (\det g) = -\varphi_{\bar{k}} \partial_{\bar{j}} (\bar{\theta}^{\bar{j}\bar{k}} (\det g)) + \partial_{\bar{j}} [\varphi_{\bar{k}} (\bar{\theta}^{\bar{j}\bar{k}} (\det g))].$$

Let $W^{\bar{k}} = (\det g)^{-1} \partial_{\bar{j}} ((\det g) \bar{\theta}^{\bar{j}\bar{k}})$. Multiplying by $i dz^1 \wedge d\bar{z}^1 \cdots \wedge i dz^n \wedge d\bar{z}^n$ and using (1.20) gives

$$\langle \bar{\partial}\varphi, \theta \rangle \frac{\omega^n}{n!} = -\varphi_{\bar{k}} W^{\bar{k}} \frac{\omega^n}{n!} + d\iota_V \frac{\omega^n}{n!}.$$

The last term involves $V^{\bar{j}} = \varphi_{\bar{k}} \theta^{\bar{j}\bar{k}}$ and will be explained below. Integrating this identity over X and applying Stokes's theorem gives

$$(\bar{\partial}\varphi, \theta)_{L^2} = -\int_X (\varphi_{\bar{k}} W^{\bar{k}}) \frac{\omega^n}{n!}.$$

We compare this with the definition of the adjoint:

$$(\bar{\partial}\varphi, \theta)_{L^2} = \int_X (\varphi_{\bar{k}} \overline{(\bar{\partial}^\dagger \theta)}^{\bar{k}}) \frac{\omega^n}{n!} = (\varphi, \bar{\partial}^\dagger \theta)_{L^2}.$$

Thus $(\bar{\partial}^\dagger \theta)^k = -\bar{W}^k$, and since

$$\bar{W}^k = (\det g)^{-1} \partial_p ((\det g) \theta^{pk})$$

this is the formula (2.1). We now explain why

$$\partial_{\bar{j}} [V^{\bar{j}} (\det g)] \, i dz^1 \wedge d\bar{z}^1 \cdots \wedge i dz^n \wedge d\bar{z}^n = d\iota_V \frac{\omega^n}{n!}.$$

First, we recall the definition of the interior product: if V is a vector field, then $\iota_V : \Lambda^k \to \Lambda^{k-1}$ via

$$(\iota_V \eta)(W_1, \ldots, W_{k-1}) = V^i \eta(\partial_i, W_1, \ldots, W_{k-1}),$$

and it satisfies $\iota_V(\eta_1 \wedge \eta_2) = \iota_V \eta_1 \wedge \eta_2 + (-1)^k \eta_1 \wedge \iota_V \eta_2$ if $\eta_1 \in \Lambda^k$. Therefore applying ι_V to (1.20) gives

$$-\iota_V \frac{\omega^n}{n!} = \det g \Big[V^{\bar{1}} (i dz^1) \wedge (i dz^2 \wedge d\bar{z}^2) \wedge (\cdots) + V^{\bar{2}} (i dz^1 \wedge d\bar{z}^1) \wedge (i dz^2) \wedge (\cdots) + \ldots \Big]$$

and $d = \partial + \bar{\partial}$ becomes

$$d\iota_V \frac{\omega^n}{n!} = \partial_{\bar{k}} [(\det g) V^{\bar{k}}] \, (i dz^1 \wedge d\bar{z}^1) \wedge \cdots \wedge (i dz^n \wedge d\bar{z}^n),$$

as claimed.

The $\bar{\partial}$-Laplacian can be studied using techniques from the theory of elliptic PDE. We now give the general definition of an elliptic operator.

Definition 2.1. *Let $E, F \to X$ be vector bundles trivialized by a finite cover $X = \cup_{i=1}^N U_i$. An elliptic operator of order k is a map $L : \Gamma(E) \to \Gamma(F)$ such that:*

- *In each trivialization $U_i \subset X$, L appears as*

$$(Lu)^\alpha = \sum_{|I|=k} A^{I\alpha}{}_\beta \partial_I u^\beta + \sum_{0 \leqslant |I| < k} B^{I\alpha}{}_\beta \partial_I u^\beta.$$

- *Let $p \in X$. Then for all $\xi \in \Lambda^1(X, \mathbb{R})$ with $\xi_p \neq 0$, then*

$$\sigma(L, \xi)(p) : E_p \to F_p$$

is an isomorphism. Here for $\xi \in \Lambda^1(X, \mathbb{R})$ with $\xi = \xi_i dx^i$, we define $\sigma(L, \xi) \in \Gamma(\mathrm{Hom}(E, F))$ by

$$\sigma(\xi)^\alpha{}_\beta = \xi_I A^{I\alpha}{}_\beta, \quad \xi_I = \xi_{i_1} \cdots \xi_{i_k}.$$

In other words, for an elliptic operator the matrix $[\xi_I A^I]$ is invertible.

Example 2.2. The $\bar{\partial}$-Laplacian $\Delta_{\bar{\partial}} : \Lambda^{p,q}(X) \to \Lambda^{p,q}(X)$ is an elliptic operator of order 2. A calculation (illustrated below) shows that it is locally of the form

$$(\Delta \psi)_{P\bar{Q}} = -g^{\bar{j}i} \partial_i \partial_{\bar{j}} \psi_{P\bar{Q}} + \ldots, \tag{2.2}$$

and therefore

$$\sigma(\Delta, \xi) = (-g^{\bar{j}i} \xi_i \xi_{\bar{j}})\, id = -|\xi|_g^2\, id.$$

Here we write $\xi = \xi_i dz^i + \xi_{\bar{i}} d\bar{z}^i$ for $\xi \in \Lambda^1(M, \mathbb{R})$, and since ξ is real then $\bar{\xi} = \xi$ and $\xi_{\bar{j}} = \overline{\xi_j}$.

We verify (2.2) for $(0,1)$ forms $\varphi = \varphi_{\bar{i}} d\bar{z}^i$. By using the expression for the adjoint (2.1) and $(\bar{\partial}\varphi)_{\bar{j}\bar{k}} = \partial_{\bar{j}} \varphi_{\bar{k}} - \partial_{\bar{k}} \varphi_{\bar{j}}$, we compute

$$\begin{aligned} (\bar{\partial}\bar{\partial}^\dagger \varphi)_{\bar{\alpha}} &= \partial_{\bar{\alpha}} (\bar{\partial}^\dagger \varphi) \\ &= -\partial_{\bar{\alpha}} \bigg((\det g)^{-1} \partial_p \Big[g^{\bar{q}p} (\det g) \varphi_{\bar{q}} \Big] \bigg) \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} (\bar{\partial}^\dagger \bar{\partial} \varphi)_{\bar{\alpha}} &= -(\det g)^{-1} \partial_p \Big[(\det g) (\bar{\partial}\varphi)^{p\beta} \Big] g_{\beta\bar{\alpha}} \\ &= -(\det g)^{-1} \partial_p \Big[(g^{\bar{q}p} g^{\bar{\nu}\beta} \det g)(\partial_{\bar{q}} \varphi_{\bar{\nu}} - \partial_{\bar{\nu}} \varphi_{\bar{q}}) \Big] g_{\beta\bar{\alpha}}. \end{aligned} \tag{2.4}$$

The terms involving 2 derivatives of φ are

$$(\Delta \varphi)_{\bar{\alpha}} = -g^{\bar{q}p} \partial_{\bar{\alpha}} \partial_p \varphi_{\bar{q}} - g^{\bar{q}p} \partial_p (\partial_{\bar{q}} \varphi_{\bar{\alpha}} - \partial_{\bar{\alpha}} \varphi_{\bar{q}}) + \ldots$$

and cancellation gives the result (2.2).

We say that an elliptic operator L is self-adjoint if

$$(L\psi, \varphi)_{L^2} = (\psi, L\varphi)_{L^2}, \quad \psi \in \Gamma(E),\ \varphi \in \Gamma(F).$$

For example, $\Delta_{\bar\partial}$ is self-adjoint.

Theorem 2.3. *Let L be a self-adjoint elliptic operator on a vector bundle over a compact manifold. There is an L^2 orthogonal decomposition*

$$\Gamma(E) = \ker L \oplus \operatorname{Im} L.$$

Thus we can solve $L\psi = \varphi$ if and only if $\varphi \in (\ker L)^\perp$.

Proof. See Theorem 7.3 in the appendix of [19]. $\square$

Applying this to $\Delta_{\bar\partial}$, we see that we can write any $\eta \in \Lambda^{p,q}(X)$ as

$$\eta = h + (\bar\partial\bar\partial^\dagger + \bar\partial^\dagger\bar\partial)\beta,$$

where $h \in \ker \Delta_{\bar\partial}$, and this is usually written as

$$\eta = h + \bar\partial\beta_1 + \bar\partial^\dagger\beta_2.$$

We will often use that

$$\ker \Delta_{\bar\partial} = \{\eta : \bar\partial\eta = 0,\ \bar\partial^\dagger\eta = 0\}.$$

This can be seen by the formula

$$(\Delta_{\bar\partial}\eta, \eta)_{L^2} = (\bar\partial\eta, \bar\partial\eta)_{L^2} + (\bar\partial^\dagger\eta, \bar\partial^\dagger\eta)_{L^2}.$$

As a consequence of this discussion, we obtain:

Corollary 2.4. *Let X be a compact complex manifold with hermitian metric $g_{i\bar j}$. Every Dolbeault cohomology class $[\eta] \in H^{p,q}(X)$ admits a unique representative $h \in [\eta]$ with $\Delta_{\bar\partial}h = 0$. Therefore:*

$$\dim\ker \Delta_{\bar\partial}|_{\Lambda^{p,q}} = \dim H^q(X, \Omega^p) = h^{p,q}.$$

Proof. Write $\eta = h + \bar\partial\beta_1 + \bar\partial^\dagger\beta_2$. Since $\bar\partial\eta = 0$, then

$$\eta = h + \bar\partial\beta_1.$$

This is because $(\bar\partial^\dagger\beta_2, \bar\partial^\dagger\beta_2) = (\beta_2, \bar\partial\eta) = 0$. It follows that $[\eta] = [h]$ for $h \in \ker \Delta_{\bar\partial}$. For uniqueness, suppose $\eta = h_1 + \bar\partial\beta_1 = \tilde h_1 + \bar\partial\tilde\beta_1$. Then

$$0 = (h_1 - \tilde h_1) + \bar\partial(\beta_1 - \tilde\beta_1)$$

and so

$$0 = (h_1 - \tilde h_1, h_1 - \tilde{h_1})_{L^2} + (\bar\partial(\beta_1 - \tilde\beta_1), h_1 - \tilde{h_1})_{L^2}$$

and $\|h_1 - \tilde h_1\|^2_{L^2} = 0$ since $\bar\partial^\dagger(h_1 - \tilde h_1) = 0$. $\square$

There is a similar theory for vector bundle valued (p,q) forms, and in general

$$\begin{aligned} H^q(X, E \otimes \Omega^p) &:= H^{p,q}_{\bar\partial}(E) \\ &= \{\eta \in \Lambda^{p,q}(E) : \Delta_{\bar\partial}\eta = 0\} \\ &= \{\eta \in \Lambda^{p,q}(E) : \bar\partial\eta = 0, \quad \bar\partial^\dagger\eta = 0\}. \end{aligned} \tag{2.5}$$

Theorem 2.5. *(Serre duality) Let $E \to X$ be a holomorphic vector bundle over a compact complex manifold. Then*

$$\dim H^{p,q}_{\bar\partial}(E) = \dim H^{n-p,n-q}_{\bar\partial}(E^*),$$

which implies that

$$\dim H^q(X, E) = \dim H^{n-q}(X, K_X \otimes E^*). \tag{2.6}$$

and $h^{p,q}(X) = h^{n-p,n-q}(X)$.

Proof. We will use the Hodge star operator. This is a linear map

$$\star : \Lambda^{p,q} \to \Lambda^{n-q,n-p}$$

defined by the property

$$\varphi \wedge \star\bar\psi = \langle \varphi, \psi \rangle_g \frac{\omega^n}{n!}, \quad \varphi, \psi \in \Lambda^k. \tag{2.7}$$

Here $\langle \varphi, \psi \rangle$ is defined by zero if φ, ψ are of different (p,q) type. For example, if g is the Euclidean metric on $\mathbb{C}^n$,

$$dz^1 \wedge \star dz^1 = \langle dz^1, d\bar z^1 \rangle_g \frac{\omega^n}{n!} = 0, \quad d\bar z^1 \wedge \star dz^1 = \frac{\omega^n}{n!}, \tag{2.8}$$

and so $\star dz^1 = -i dz^1 \wedge (i dz^2 \wedge d\bar z^2) \wedge \cdots \wedge (i dz^n \wedge d\bar z^n)$.

The Hodge star satisfies:

- $\overline{\star\psi} = \star\bar\psi$
- $\star^2 \psi^{p,q} = (-1)^{p+q}\psi^{p,q}$

The first property can be verified by manipulating and taking the conjugate of (2.7). The second property can be verified by manipulating (2.7) and using $\langle \star\varphi, \star\psi \rangle = \langle \varphi, \psi \rangle$. To show $\star$ preserves the inner product, one can calculate in a similar way to (2.8) to show that $\star$ takes an orthonormal basis of $\Lambda^{p,q}$ to an orthonormal basis $\Lambda^{n-q,n-p}$.

Next, we extend $\star$ to vector bundle valued forms. For $\varphi \in \Lambda^{p,q}(E)$, we can write $\varphi = \varphi^\alpha \otimes \eta^{p,q}$ and define

$$\star(\varphi^\alpha \otimes \eta^{p,q}) = \varphi^\alpha \otimes \star\eta^{p,q}.$$

Equip E with a hermitian metric H. The L^2 inner product can then be written as

$$(\varphi, \psi)_{L^2} = \int \varphi^\alpha \wedge \star\overline{\psi^\beta} H_{\alpha\bar\beta}.$$

The L^2 adjoint of $\bar\partial$ can be written as

$$(\bar\partial^\dagger \psi)^\beta = - \star H^{\bar\mu\beta} \partial (H_{\nu\bar\mu} \star \psi^\nu). \tag{2.9}$$

This can be verified by Stokes's theorem and substitution of (2.9) into the defining relation $(\bar\partial\varphi,\psi) = (\varphi,\bar\partial^\dagger\psi)$. Indeed, for $\varphi \in \Lambda^{p,q}(E)$ and $\psi \in \Lambda^{p,q+1}(E)$, then

$$\begin{aligned}
(\varphi,\bar\partial^\dagger\psi) &= \int \varphi^\alpha \wedge \star\overline{(\bar\partial^\dagger\psi)^\beta} H_{\alpha\bar\beta} \\
&= -\int \varphi^\alpha \wedge \star^2\Big(H^{\bar\beta\mu}\bar\partial(H_{\mu\bar\nu}\star\bar\psi^{\bar\nu})\Big)H_{\alpha\bar\beta} \\
&= (-1)(-1)^{n-p}(-1)^{n-q}\int \varphi^\alpha \wedge \bar\partial(H_{\alpha\bar\nu}\star\bar\psi^{\bar\nu}) \\
&= (-1)(-1)^{n-p}(-1)^{n-q}(-1)(-1)^{p+q}\int \bar\partial\varphi^\alpha \wedge \star\bar\psi^{\bar\nu}H_{\alpha\bar\nu} \\
&= (\bar\partial\varphi,\psi).
\end{aligned} \tag{2.10}$$

Next, we define the map

$$\# : \Lambda^{p,q}(E) \to \Lambda^{n-p,n-q}(E^*)$$

by

$$(\#\psi)_\beta = H_{\beta\bar\alpha}\star\overline{\psi^\alpha}.$$

This satisfies $\#\# = (-1)^{p+q}$, and formula (2.9) can be written

$$\overline{(\bar\partial^\dagger\psi)^\mu} = -\star\bar\partial(\#\psi)_\beta H^{\beta\bar\mu}.$$

Therefore

$$H^{p,q}_{\bar\partial}(E) = \{\eta \in \Lambda^{p,q}(E) : \bar\partial\eta = 0, \quad \bar\partial^\dagger\eta = 0\}$$

can be written in this notation as

$$H^{p,q}_{\bar\partial}(E) = \{\eta \in \Lambda^{p,q}(E) : \bar\partial\eta = 0, \quad \bar\partial\#\eta = 0\}.$$

It follows that

$$\eta \mapsto \#\eta$$

is a map from $H^{p,q}(E)$ to $H^{n-p,n-q}(E^*)$, and this is an isomorphism.

Note that here we used $\#$ because this would not have worked using $\star : \Lambda^{p,q}(X) \to \Lambda^{n-q,n-p}(X)$, since $\Delta_{\bar\partial}\eta = 0$ if and only if $\bar\partial\eta = 0$ and $\partial\star\eta = 0$, and so if $\Delta_{\bar\partial}\eta = 0$ then it is not necessarily true that $\Delta_{\bar\partial}\star\eta = 0$.

$\square$

On a Kähler manifold, there are the following symmetries for the Hodge numbers.

Theorem 2.6. *Let X be a compact Kähler manifold. Then*

$$H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X), \quad h^{p,q} = h^{q,p}, \quad h^{n-p,n-q} = h^{p,q}.$$

If $b_k = \dim H^k(X,\mathbb{C})$ denote the Betti numbers of X, then

$$b_k = \sum_{p=0}^{k} h^{p,k-p}. \tag{2.11}$$

For example,

$$\begin{aligned} b_1 &= h^{1,0} + h^{0,1} = 2h^{0,1}, \\ b_2 &= h^{2,0} + h^{1,1} + h^{0,2} = 2h^{0,2} + h^{1,1}. \end{aligned} \tag{2.12}$$

Here

$$H^q(X,\mathbb{C}) = \frac{\ker(d : \Lambda^q(X) \to \Lambda^{q+1}(X))}{\mathrm{im}\,(d : \Lambda^{q-1}(X) \to \Lambda^q(X))}$$

are the deRham cohomology groups, and $\Lambda^k(X)$ denotes differential k-forms with coefficients in $\mathbb{C}$. The Laplacians are

$$\Delta_d = dd^\dagger + d^\dagger d, \quad \Delta_\partial = \overline{\Delta_{\bar\partial}} = \partial\partial^\dagger + \partial^\dagger\partial,$$

where $d^\dagger$, $\partial^\dagger$ are the L^2 adjoints of d, ∂. These Laplacians are elliptic operators and satisfy the Hodge decomposition Theorem 2.3, and it follows from elliptic PDE theory that each de Rham class $[\eta] \in H^k(X,\mathbb{C})$ admits a unique representative $h \in [\eta]$ with $\Delta_d h = 0$.

Proof. This theorem follows from the Kähler Laplacian identities

$$\Delta_{\bar\partial} = \Delta_\partial = \frac{1}{2}\Delta_d. \tag{2.13}$$

We will prove (2.13) below. We will show how (2.13) implies the result. From

$$\begin{aligned} H^k(X,\mathbb{C}) &= \{\eta \in \Lambda^k : \Delta_d \eta = 0\} \\ H^{p,q}(X) &= \{\eta \in \Lambda^{p,q} : \Delta_{\bar\partial}\eta = 0\}, \end{aligned} \tag{2.14}$$

we can decompose $\eta \in \Lambda^k$ as $\eta = \sum_{p+q=k} \eta^{p,q}$. Since $\Delta_{\bar\partial}$ preserves type, we have $\Delta_d \eta = \sum_{p+q=k} 2\Delta_{\bar\partial}\eta^{p,q}$. Therefore if $\eta \in \Lambda^k$ with $\Delta_d \eta = 0$, then

$$\eta \mapsto \sum_{p+q=k} \eta^{p,q}$$

is a map from $H^k(X,\mathbb{C})$ to $\oplus_{p+q=k} H^{p,q}(X)$ and this is an isomorphism.

Next, we prove $h^{p,q} = h^{q,p}$. If $\eta \in H^{p,q}(X)$ with $\Delta_{\bar\partial}\eta = 0$, then $\bar\eta \in \Lambda^{q,p}(X)$ and (2.13) implies

$$\Delta_{\bar\partial}\bar\eta = \overline{\Delta_{\partial}}\bar\eta = \overline{\Delta_{\partial}\bar{\bar\eta}} = 0.$$

Therefore $\eta \mapsto \bar\eta$ is a map from $H^{p,q}(X)$ to $H^{q,p}(X)$ and this is an isomorphism. □

Example 2.7. We compute the Hodge numbers of $\mathbb{P}^n$. The cell-decomposition $\mathbb{P}^n = \{pt\} \cup \mathbb{C}^1 \cup \mathbb{C}^2 \cup \cdots \cup \mathbb{C}^n$ implies

$$\dim H^{2k}(\mathbb{P}^n,\mathbb{C}) = 1, \quad k \in \{0,1,\dots\}$$

and all odd cohomology groups are zero. Since $H^k(X,\mathbb{C}) = \bigoplus H^{p,q}(X)$, we conclude

$$h^{k,k}(\mathbb{P}^n) = 1$$

and all other Hodge numbers are zero.

We now show $\Delta_{\bar\partial} = \Delta_\partial = \frac{1}{2}\Delta_d$. First, we note the following Kähler identities:

$$(\bar\partial^\dagger \varphi)_{J\bar K} = -g^{\bar a b}\nabla_b \varphi_{\bar a J \bar K}, \tag{2.15}$$

and

$$\begin{aligned} i\bar\partial^\dagger &= \partial\Lambda_\omega - \Lambda_\omega \partial \\ -i\partial^\dagger &= \bar\partial\Lambda_\omega - \Lambda_\omega\bar\partial, \end{aligned} \tag{2.16}$$

where $\Lambda_\omega : \Lambda^{p,q} \to \Lambda^{p-1,q-1}$ is

$$(\Lambda\varphi)_{J\bar K} = i g^{\bar a b}\varphi_{b\bar a I \bar J}.$$

For example, for $\omega = i g_{j\bar k} dz^j \wedge d\bar z^k$ then $\Lambda_\omega \omega = -n$.

We will discuss (2.15) in the following section on the Kodaira vanishing theorem, and we assume it for now. Let us verify (2.16) for $\varphi \in \Lambda^{1,1}$. Then

$$(\partial\Lambda_\omega\varphi)_j = \nabla_j(i g^{\bar a b}\varphi_{b\bar a}) = i g^{\bar a b}\nabla_j \varphi_{b\bar a}.$$

and

$$(\Lambda_\omega \partial\varphi)_j = i g^{\bar a b}(\partial\varphi)_{b\bar a j},$$

while

$$\begin{aligned} \partial\varphi &= \partial_\ell \varphi_{j\bar k} dz^\ell \wedge dz^j \wedge d\bar z^k \\ &= \frac{1}{2}(\nabla_\ell \varphi_{j\bar k} - \nabla_j \varphi_{\ell\bar k}) dz^\ell \wedge dz^j \wedge d\bar z^k \\ &= \frac{1}{2}(\partial\varphi)_{\ell j\bar k} dz^\ell \wedge dz^j \wedge d\bar z^k \end{aligned} \tag{2.17}$$

where ∂ was switched for ∇ since $\Gamma^k_{ij} = \Gamma^k_{ji}$ for Kähler metrics, as is seen by the explicit formula $\Gamma^k_{ij} = \partial_i g_{j\bar p} g^{\bar p k}$ and the Kähler definition $\partial_i g_{j\bar p} = \partial_j g_{i\bar p}$. So

$$(\Lambda_\omega\partial\varphi)_j = i g^{\bar a b}(-\nabla_b\varphi_{j\bar a} + \nabla_j \varphi_{b\bar a})$$

Therefore

$$(\partial\Lambda_\omega\varphi)_j - (\Lambda_\omega\partial\varphi)_j = i g^{\bar a b}\nabla_b \varphi_{j\bar a} = i(\bar\partial^\dagger\varphi)_j$$

which proves (2.16) for $(1,1)$-forms.

Next, using the Kähler identity (2.16), we note

$$\partial\bar\partial^\dagger = -\bar\partial^\dagger\partial, \tag{2.18}$$

since

$$i\partial\bar\partial^\dagger + i\bar\partial^\dagger\partial = \partial(\partial\Lambda - \Lambda\partial) + (\partial\Lambda - \Lambda\partial)\partial = 0.$$

Combining (2.16) and (2.18), we derive

$$\begin{aligned} \Delta_\partial &= (\partial\partial^\dagger + \partial^\dagger\partial) \\ &= i\partial(\bar\partial\Lambda - \Lambda\bar\partial) + i(\bar\partial\Lambda - \Lambda\bar\partial)\partial \\ &= -i\bar\partial(\partial\Lambda - \Lambda\partial) - i(\partial\Lambda - \Lambda\partial)\bar\partial \\ &= \bar\partial\bar\partial^\dagger + \bar\partial^\dagger\bar\partial \\ &= \Delta_{\bar\partial} \end{aligned} \tag{2.19}$$

and

$$
\begin{aligned}
\Delta_d &= (dd^\dagger + d^\dagger d) \\
&= (\partial + \bar\partial)(\partial^\dagger + \bar\partial^\dagger) + (\partial^\dagger + \bar\partial^\dagger)(\partial + \bar\partial) \\
&= \Delta_\partial + \Delta_{\bar\partial} + (\partial\bar\partial^\dagger + \bar\partial^\dagger\partial) + (\bar\partial\partial^\dagger + \partial^\dagger\bar\partial) \\
&= 2\Delta_{\bar\partial}.
\end{aligned} \tag{2.20}
$$

To end this section, we state the $\partial\bar\partial$-Lemma.

Theorem 2.8. *($\partial\bar\partial$-Lemma) Let X be a compact Kähler manifold. Let $\alpha \in \Lambda^{p,q}(X)$. Then:*

- *If $\alpha = d\eta$ for $\eta \in \Lambda^{p+q-1}$, then $\alpha = \partial\bar\partial\beta$ for $\beta \in \Lambda^{p-1,q-1}$.*
- *If $\alpha = \partial\eta$ for $\eta \in \Lambda^{p-1,q}$ and $d\alpha = 0$, then $\alpha = \partial\bar\partial\beta$ for $\beta \in \Lambda^{p-1,q-1}$.*

Proof. We prove the first statement, and the second statement has a similar proof. Using the Hodge decomposition (Theorem 2.3) for Δ_∂,

$$\alpha = \partial\alpha_1 + \partial^\dagger\alpha_2 + \alpha', \quad \alpha' \in \ker\Delta_\partial.$$

Using that $\alpha' \in \ker\Delta_\partial = \ker\Delta_d$, we have $(\alpha', \alpha') = (\alpha', \alpha) = (d^\dagger\alpha', \eta) = 0$. Also $0 = \partial\alpha = \partial\partial^\dagger\alpha_2$ and so $(\partial^\dagger\alpha, \partial^\dagger\alpha) = 0$. Therefore

$$\alpha = \partial\alpha_1.$$

Next, we use the Hodge decomposition for $\Delta_{\bar\partial}$ to write

$$\alpha_1 = \bar\partial\beta_1 + \bar\partial^\dagger\beta_2 + \beta', \quad \beta' \in \ker\Delta_{\bar\partial}$$

Substituting gives

$$\alpha = \partial\bar\partial\beta_1 + \partial\bar\partial^\dagger\beta_2.$$

To remove the last term, we must use the Kähler identity (2.18) which reads $\partial\bar\partial^\dagger = -\bar\partial^\dagger\partial$. Using this and $\bar\partial\alpha = 0$, we see that

$$0 = \bar\partial\bar\partial^\dagger\partial\beta_2.$$

It follows that $(\bar\partial^\dagger\partial\beta_2, \bar\partial^\dagger\partial\beta_2) = 0$.

□

2.2 Kodaira vanishing theorem

Let $L \to (M, \omega)$ be a holomorphic line bundle over a compact Kähler manifold. We say that L is a positive line bundle if it admits a metric h such that its curvature satisfies $F_{j\bar k} \geqslant \varepsilon g_{j\bar k}$ for some $\varepsilon > 0$. Here the inequality is in the sense of positive-definite matrices, meaning

$$F_{j\bar k} v^j \overline{v^k} \geqslant \varepsilon g_{j\bar k} v^j \overline{v^k}$$

for all $v \in \mathbb{C}^n$.

Example 2.9. The main example of a positive line bundle is $\mathcal{O}(1) \to \mathbb{P}^n$, as

$$(h_{FS})_i = \frac{|Z_i|^2}{\sum_p |Z_p|^2}, \quad \text{over } U_i = \{Z_i \neq 0\}$$

defines a metric with curvature $iF = -i\partial\bar{\partial}\log h_{FS} = \omega_{FS} > 0$ as we computed in (1.26). A similar computation shows that $\mathcal{O}(k) \to \mathbb{P}^n$ is positive for any $k \geqslant 1$. For this, equip $\mathcal{O}(k)$ with $(h_{FS})^k$ so that the curvature is $iF = k\omega_{FS}$.

Example 2.10. Let $P(Z_0, \ldots, Z_n)$ be a homogeneous polynomial of degree k, and let $Y = \{P = 0\} \subset \mathbb{P}^n$. Then $\mathcal{O}(Y) \to \mathbb{P}^n$ is a positive line bundle by example 1.14.

Example 2.11. There is a notion of ample line bundle $L \to M$, which means that there are sections $s_i \in H^0(M, L)$ such that $\varphi : M \to \mathbb{P}^N$ with $\varphi(z) = [s_0(z), \ldots, s_N(z)]$ is an embedding. We can cover X with trivializations $U_i = \{s_i \neq 0\}$ and equip L with metrics (h_i, U_i) with

$$h_i = \varphi^* h_{FS} = \frac{|s_i|^2}{\sum_k |s_k|^2}.$$

Then $-i\partial\bar{\partial}\log h = \varphi^*(-i\partial\bar{\partial}\log h_{FS}) > 0$. Therefore ample line bundles are positive. Kodaira's embedding theorem (e.g. [20]) states that positive line bundles are ample.

Theorem 2.12. *Let $L \to (M, \omega)$ be a positive holomorphic line bundle over a compact Kähler manifold. Then*

$$H^q(X, L \otimes K_X) = 0$$

for all $q \geqslant 1$.

We will show that dim $H^q(X, L \otimes K_X) = \dim\ker \Delta_{\bar{\partial}}|_{\Lambda^{0,q}(L\otimes K_X)} = 0$. We will give the proof for $q = 1$ for simplicity. For the general calculation, see e.g. [20].

Let h be a metric on L, so that the inner product on sections $u, v \in \Gamma(L)$ is $\langle u, v\rangle_h = u\bar{v}h$. Let $\varphi \in \Lambda^{0,1}(L)$, which we write as

$$\varphi = \varphi_{\bar{k}} d\bar{z}^k,$$

where $\varphi_{\bar{k}}$ is a local section of L. The L^2 inner product for $u, v \in \Gamma(L)$ is

$$(u, v) = \int_X (u\bar{v}h)\, \frac{\omega^n}{n!}$$

and for $\varphi, \psi \in \Lambda^{0,1}(L)$ is

$$(\varphi, \psi) = \int_X g^{\bar{i}k}(\varphi_{\bar{i}}\overline{\psi_{\bar{k}}}h)\, \frac{\omega^n}{n!}.$$

We will start by computing $\bar{\partial}^\dagger\varphi \in \Gamma(L)$. The difference with (2.1), in addition to introducing the line bundle L into the mix, is that we now use the assumption that g is Kähler. The formula in this case is:

Lemma 2.13. *Let (X, ω) be a compact Kähler manifold and $\varphi \in \Lambda^{0,q}(L)$.*

$$(\bar{\partial}^\dagger\varphi)_{\bar{K}} = -g^{\bar{a}b}\nabla_b\varphi_{\bar{a}\bar{K}}.$$

We will verify this for $\varphi \in \Lambda^{0,1}(L)$. The definition of the adjoint is

$$(\bar{\partial}^\dagger \varphi, u) = (\varphi, \bar{\partial} u), \quad \varphi \in \Lambda^{0,1}(L),\ u \in \Gamma(L).$$

We start with

$$\langle \varphi, \bar{\partial} u \rangle_{g,h} = g^{\bar{i}k} (\varphi_{\bar{i}} \overline{\partial_{\bar{k}} u} h) = \varphi^k \partial_k \overline{u} h$$

which implies

$$\langle \varphi, \bar{\partial} u \rangle_{g,h} (\det g) = \partial_k (\varphi^k \overline{u} (\det g) h) - \partial_k (h (\det g) \varphi^k) \overline{u}$$

The first term integrates to zero by Stokes's theorem (see earlier notes for more justification on this), and so wedging by $dz^1 \wedge \cdots \wedge d\bar{z}^n$ and integrating gives

$$(\varphi, \bar{\partial} u) = -\int_X \partial_k (h (\det g) \varphi^k) (\det g)^{-1} h^{-1} h \overline{u} \, \frac{\omega^n}{n!} = (\bar{\partial}^\dagger \varphi, u)$$

where

$$\bar{\partial}^\dagger \varphi = -h^{-1} (\det g)^{-1} \partial_k (h (\det g) g^{k\bar{i}} \varphi_{\bar{i}}). \tag{2.21}$$

On a Kähler manifold, this is in fact

$$\bar{\partial}^\dagger \varphi = -g^{\bar{k}p} \nabla_p \varphi_{\bar{k}}, \tag{2.22}$$

where the Chern connection of (h, g) acts on $\varphi \in \Lambda^{0,1}(L)$ by

$$\nabla_p \varphi_{\bar{k}} = \partial_p \varphi_{\bar{k}} + (h^{-1} \partial_p h) \varphi_{\bar{k}}, \quad \nabla_{\bar{p}} \varphi_{\bar{k}} = \partial_{\bar{p}} \varphi_{\bar{k}} - \Gamma_{\bar{p}\bar{k}}{}^{\bar{\ell}} \varphi_{\bar{\ell}}.$$

To see (2.22), expand (2.21)

$$\bar{\partial}^\dagger \varphi = -g^{k\bar{i}} \partial_k \varphi_{\bar{i}} - h^{-1} \partial_k h g^{k\bar{i}} \varphi_{\bar{i}} - (\det g)^{-1} \partial_k ((\det g) g^{k\bar{i}}) \varphi_{\bar{i}}$$

The last term is zero. Indeed,

$$\begin{aligned} \partial_k ((\det g) g^{\bar{i}k}) &= \partial_k (\det g) g^{\bar{i}k} + (\det g) \partial_k g^{\bar{i}k} \\ &= (\det g) g^{\bar{b}a} \partial_k g_{a\bar{b}} g^{\bar{i}k} - (\det g) g^{\bar{b}k} \partial_k g_{a\bar{b}} g^{\bar{i}a} \\ &= 0 \end{aligned} \tag{2.23}$$

by the Kähler condition $\partial_k g_{a\bar{b}} = \partial_a g_{k\bar{b}}$. This proves (2.22). With this formula for the adjoint, we now compute the Laplacian.

Lemma 2.14. *Let $(L, h) \to (X, \omega)$ be a holomorphic line bundle with metric over a compact Kähler manifold. For any $\varphi \in \Lambda^{0,1}(L)$, we have*

$$(\Delta_{\bar{\partial}} \varphi)_{\bar{k}} = -g^{\bar{a}b} \nabla_b \nabla_{\bar{a}} \varphi_{\bar{k}} + \varphi^i R_{i\bar{k}} + \varphi^i F_{i\bar{k}}.$$

Proof. We start with $(\bar{\partial} f)_{\bar{k}} = \nabla_{\bar{k}} f$ on functions, which implies

$$(\bar{\partial} \bar{\partial}^\dagger \varphi)_{\bar{k}} = \nabla_{\bar{k}} (-g^{\bar{a}b} \nabla_b \varphi_{\bar{a}}) = -g^{\bar{a}b} \nabla_{\bar{k}} \nabla_b \varphi_{\bar{a}},$$

since $\nabla_{\bar{k}} g^{\bar{a}b} = 0$. Next,

$$
\begin{aligned}
(\bar{\partial}^\dagger \bar{\partial} \varphi)_{\bar{k}} &= -g^{\bar{a}b} \nabla_b (\bar{\partial}\varphi)_{\bar{a}\bar{k}} \\
&= -g^{\bar{a}b} \nabla_b (\partial_{\bar{a}} \varphi_{\bar{k}} - \partial_{\bar{k}} \varphi_{\bar{a}}) \\
&= -g^{\bar{a}b} \nabla_b (\nabla_{\bar{a}} \varphi_{\bar{k}} - \nabla_{\bar{k}} \varphi_{\bar{a}}).
\end{aligned}
\tag{2.24}
$$

This is because ${\Gamma_{ij}}^k = {\Gamma_{ji}}^k$ for a Kähler manifold since ${\Gamma_{ij}}^k = \partial_i g_{j\bar{p}} g^{\bar{p}k}$. Therefore

$$
(\Delta_{\bar{\partial}} \varphi)_{\bar{k}} = -g^{\bar{a}b} \nabla_b \nabla_{\bar{a}} \varphi_{\bar{k}} + [\nabla_b, \nabla_{\bar{k}}] \varphi^b
$$

The commutator formula for covariant derivatives on $\varphi^a \in \Gamma(T^{1,0}X \otimes L)$ implies

$$
[\nabla_b, \nabla_{\bar{k}}] \varphi^b = \varphi^a {R_a}^b{}_{b\bar{k}} + \varphi^b F_{b\bar{k}}.
$$

In Kähler geometry, one can see directly from ${R_j}^i{}_{m\bar{k}} = -\partial_{\bar{k}} (\partial_m g_{j\bar{p}} g^{\bar{p}i})$ the symmetry

$$
{R_a}^b{}_{j\bar{k}} = {R_j}^b{}_{a\bar{k}}.
$$

Since $R_{a\bar{k}} = {R_b}^b{}_{a\bar{k}}$, we conclude the formula. □

We now let $\varphi \in \Lambda^{0,1}(L \otimes K_X)$. We can apply the previous formula with $\varphi \in \Lambda^{0,1}(\tilde{L})$, and $\tilde{L} = L \otimes K_X$ equipped with the product metric $\tilde{h} = h \otimes (\det g)^{-1}$. Using the formula $R_{i\bar{k}} = -\partial_i \partial_{\bar{k}} \log \det g$ and $\tilde{F}_{j\bar{k}} = -\partial_j \partial_{\bar{k}} \log h$, we see that the curvature is

$$
\tilde{F}_{j\bar{k}} = F_{j\bar{k}} - R_{j\bar{k}}
$$

Therefore cancellation occurs and we have

$$
(\Delta_{\bar{\partial}} \varphi)_{\bar{k}} = -g^{\bar{a}b} \nabla_b \nabla_{\bar{a}} \varphi_{\bar{k}} + \varphi^i F_{i\bar{k}}.
$$

Suppose $\varphi \in \ker \Delta$. Then

$$
0 = (\Delta \varphi, \varphi) = -\int_X \nabla_a \nabla^a \varphi_{\bar{k}} \bar{\varphi}^{\bar{k}} h \frac{\omega^n}{n!} + \int_X \varphi^i F_{i\bar{k}} \bar{\varphi}^{\bar{k}} h \frac{\omega^n}{n!}
$$

Integrating the first term by parts and using positivity of $F_{i\bar{k}}$,

$$
0 \geqslant \int_X \nabla^a \varphi_{\bar{k}} \nabla_a \bar{\varphi}^{\bar{k}} h \, \frac{\omega^n}{n!} + \varepsilon \int_X g_{i\bar{k}} \varphi^i \bar{\varphi}^{\bar{k}} h \, \frac{\omega^n}{n!},
$$

which is

$$
0 \geqslant (\nabla \varphi, \nabla \varphi)_{L^2} + \varepsilon (\varphi, \varphi)_{L^2}.
$$

It follows that $\varphi = 0$. This proves that $\ker \Delta = \{0\}$. Therefore $\dim H^1(X, L \otimes K_X) = 0$.

Here we used integration by parts on a Kähler manifold, which follows from the divergence theorem

$$
\int_X \nabla_a V^a \, \omega^n = 0, \quad V \in \Gamma(T^{1,0}X).
\tag{2.25}
$$

In the above computation, this was used with $V^a = \nabla^a \varphi_{\bar{k}} \bar{\varphi}^{\bar{k}} h \in \Gamma(T^{1,0}X)$ and

$$0 = \int_X \nabla_a (\nabla^a \varphi_{\bar{k}} \bar{\varphi}^{\bar{k}} h) \, \frac{\omega^n}{n!} = \int_X \nabla_a \nabla^a \varphi_{\bar{k}} \bar{\varphi}^{\bar{k}} h \, \frac{\omega^n}{n!} + \int_X \nabla^a \varphi_{\bar{k}} \nabla_a (\bar{\varphi}^{\bar{k}} h) \, \frac{\omega^n}{n!}$$

and we also then used metric compatibility $\nabla_a h = 0$. The divergence theorem (2.25) comes from

$$\int_X d\iota_V \frac{\omega^n}{n!} = 0,$$

and the integrand is

$$\partial_i (V^i \det g)(i dz^1 d\bar{z}^1) \dots (i dz^n d\bar{z}^n) = (\det g)^{-1} \partial_i (V^i \det g) \frac{\omega^n}{n!}.$$

We compute

$$(\det g)^{-1} \partial_i (V^i \det g) = \partial_i V^i + (\det g)^{-1} \partial_i (\det g) V^i = \partial_i V^i + g^{\bar{a}b} \partial_i g_{\bar{a}b} V^i.$$

Therefore

$$\int_X (\partial_i V^i + \Gamma_{ib}{}^b V^i) \, \omega^n = 0.$$

On the other hand

$$\nabla_i V^i = \partial_i V^i + \Gamma_{ib}{}^i V^b.$$

Since $\Gamma_{ib} = \Gamma_{bi}$, these are equal.

This vanishing theorem can be generalized, and more generally there holds (see e.g. [20]

Theorem 2.15. *Let $L \to (X, \omega)$ be a positive holomorphic line bundle over a compact Kähler manifold. Then*

$$H^q(X, \Omega^p \otimes L) = 0$$

for all integers p, q with $p + q > n$.

2.3 Sheaves and the Lefschetz hyperplane theorem

We will use the vanishing theorem (Theorem 2.15) to prove the Lefschetz hyperplane theorem.

First, we state some results from the theory of sheaves. For a nonempty open set $U \subseteq X$, let $\mathcal{O}(U)$ denote holomorphic functions on U. Note that the only holomorphic functions defined on the entirety of a compact manifold are constant functions. But for small open sets $U \subseteq X$ there are many holomorphic functions in $\mathcal{O}(U)$.

Here we prove: if $f : X \to \mathbb{C}$ is holomorphic on a compact complex manifold X, then f is constant. Let $M = \sup_X |f|$ be attained a point $p \in X$. After possibly replacing f by $e^{i\theta} f$ for a constant angle $e^{i\theta}$, we may assume that $f(p) = M$. Consider $\mathrm{Re}\, f = \frac{1}{2}(f + \bar{f})$, so that $\mathrm{Re}\, f \leqslant M$ and $\mathrm{Re}\, f(p) = M$. Let $S = \{x \in X : \mathrm{Re}\, f(x) = M\}$. Then S is non-empty and closed. It is also in fact open: if $x \in S$, in a local coordinate ball $B_1(0)$ centered at x then $\mathrm{Re}\, f$ is a harmonic function:

$$\partial_{z^1} \partial_{\bar{z}^1} \mathrm{Re}\, f = \frac{1}{2} \partial_{z^1} (\overline{\partial_{z^1} f}) = 0,$$

so that $\sum(\partial_{x^i}\partial_{x^i} + \partial_{y^i}\partial_{y^i})\mathrm{Re}\, f = 0$. By the maximum principle for harmonic functions on $B_1(0) \subseteq \mathbb{R}^{2n}$, since $(\mathrm{Re}\, f)(0) = M$ then $\mathrm{Re}\, f \equiv M$ in all of $B_1(0)$. Hence S is open, and $S = X$. Since $\mathrm{Re}\, f = M$ and $|f| \leqslant M$, then $\mathrm{Im}\, f = 0$ and f is a constant.

We will define sheaves of $\mathcal{O}(U)$ modules (there is also a notion of sheaves of groups, vector spaces, etc).

Definition 2.16. *A presheaf (of $\mathcal{O}_X$ modules) $\mathcal{F}$ on a complex manifold X is defined by the following information. For any non-empty open set $U \subseteq X$, we associate a nonempty $\mathcal{O}(U)$-module $\mathcal{F}(U)$, and a collection of restriction maps $\rho_{U,V} : \mathcal{F}(V) \to \mathcal{F}(U)$ defined when $U \subseteq V$ satisfying*

$$\rho_{U,V} \circ \rho_{V,W} = \rho_{U,W}, \quad \rho_{U,U} = id_U, \tag{2.26}$$

for $U \subseteq V \subseteq W$. The set $\mathcal{F}(U)$ is called the set of sections of $\mathcal{F}$ over U. We also use the notation $s|_U = \rho_{UV}(s)$ for $s \in \mathcal{F}(V)$.

Definition 2.17. *A sheaf $\mathcal{F}$ on X is a presheaf satisfying the following glueing property. Suppose $\Omega = \bigcup U_\mu$ are open sets in X. If $s_\mu \in \mathcal{F}(U_\mu)$ are such that*

$$\rho_{U_\mu \cap U_\nu, U_\mu}(s_\mu) = \rho_{U_\mu \cap U_\nu, U_\nu}(s_\nu) \tag{2.27}$$

then there exists $s \in \mathcal{F}(\Omega)$ such that $\rho_{U_\mu,\Omega}(s) = s_\mu$. Also, if $s, t \in \mathcal{F}(\Omega)$ and $\rho_{U_\mu,\Omega}(s) = \rho_{U_\mu,\Omega}(t)$ for all μ, then $s = t$.

In other words, local sections of a sheaf can be uniquely glued together.

Example 2.18. We write $\mathcal{O}_X$ for the sheaf of holomorphic functions on a complex manifold X.

Example 2.19. Let $E \to M$ be a holomorphic bundle. We write $\mathcal{E}$ for the sheaf of holomorphic sections: $\mathcal{E}(U)$ are holomorphic sections over U.

Example 2.20. Sheaf $\mathcal{I}_0$ described by holomorphic functions in $\mathbb{C}^2$ vanishing at the origin. If U does not contain the origin, then this is generated by 1 so $\mathcal{I}_0(U) \cong \mathcal{O}(U)$. In a neighbourhood V of the origin, this is generated by x and y: any local holomorphic function f with $f(0) = 0$ can be written as $f(x) = g(x, y)x + h(x, y)y$. Thus $\mathcal{I}_0(V)$ is a module of rank 2. Thus the rank jumps up to 2. Also, at the origin, the module is not free. For example, we have the relation $-y \cdot x + x \cdot y = 0$. In this sense, sheaves are sometimes viewed as a generalization of vector bundles where the rank may jump.

Example 2.21. Another example to note are the constant sheaves $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$. These are sheaves of groups, meaning that $\mathcal{F}(U)$ attaches a group for every open set U (rather than a module). So for example, $\mathbb{Z}(U)$ are locally constant $\mathbb{Z}$-valued functions on U.

Definition 2.22. *Stalk of a sheaf. Let $x \in X$. The stalk $\mathcal{F}_x$ is the set of equivalence classes in the disjoint union $\bigsqcup_{x \in U} \mathcal{F}(U)$ with $s_1 \in \mathcal{F}(U_1)$ and $s_2 \in F(U_2)$ satisfying $s_1 \sim s_2$ if $s_1|_V = s_2|_V$ for some $V \subset U_1 \cap U_2$. The stalk $\mathcal{F}_x$ is an $\mathcal{O}_{x,X}$-module.*

A map of sheaves $\varphi : \mathcal{F} \to \mathcal{E}$ is a collection of homomorphisms $\varphi_U : \mathcal{F}(U) \to \mathcal{E}(U)$ such that φ_U, φ_V commute with the restriction maps. We say

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

is an exact sequence of sheaves if, denoting the arrows by f_i, we have that all f_i are maps of sheaves with $f_{i+1} \circ f_i = 0$ and the associated complex of stalks

$$0 \to \mathcal{E}_x \to \mathcal{F}_x \to \mathcal{G}_x \to 0$$

is exact for all $x \in X$. Recall that exact means that the kernel of one arrow is the image of the previous arrow.

We will use the following two results from Cech cohomology [13]. Rather than define the Cech cohomology groups $\check{H}^q(X, \mathcal{E})$, we will just directly use the following two facts:

- Given an exact sequence of sheaves $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$, there exists a long exact sequence in cohomology
$$\cdots \to \check{H}^p(X, \mathcal{E}) \to \check{H}^p(X, \mathcal{F}) \to \check{H}^p(X, \mathcal{G}) \to \check{H}^{p+1}(X, \mathcal{E}) \to \cdots.$$

- Dolbeault theorem:
$$\check{H}^q(X, \Omega^p \otimes \mathcal{E}) \cong \frac{\ker(\bar{\partial} : \Lambda^{p,q}(E) \to \Lambda^{p,q+1}(E))}{\mathrm{im}\,(\bar{\partial} : \Lambda^{p,q-1}(E) \to \Lambda^{p,q}(E))}.$$

We will write $H^q(X, \Omega^p \otimes E)$ as before instead of $\check{H}^q(X, \Omega^p \otimes \mathcal{E})$.

Theorem 2.23. *(Lefschetz hyperplane) Let $Y \subseteq X$ be a smooth analytic hypersurface of a compact Kähler manifold X such that the line bundle $\mathcal{O}(Y)$ is positive. Then the restriction map*

$$H^q(X, \Omega^p_X) \to H^q(Y, \Omega^p_Y)$$

is an isomorphism when $p + q \leqslant n - 2$. Thus we have equality of Hodge numbers:

$$h^{p,q}(Y) = h^{p,q}(X), \quad p + q \leqslant n - 2.$$

As a consequence of the Hodge decomposition, we obtain that

$$H^q(X, \mathbb{C}) \to H^q(Y, \mathbb{C})$$

is an isomorphism for $q \leqslant n - 2$.

Example 2.24. Let $Y = \{P = 0\} \subset \mathbb{P}^n$ be a smooth complex manifold cut out by a homogeneous polynomial P of degree $m \geqslant 1$. We showed earlier that $\mathcal{O}(Y) = \mathcal{O}(m)$, which is positive. Therefore $H^k(Y, \mathbb{C})$ is isomorphic to $H^k(\mathbb{P}^n)$ for $k \leqslant n - 2$.

Proof. First, we note the exact sequence sheaves

$$0 \to \mathcal{O}(-Y) \to \mathcal{O}_X \to \mathcal{O}_Y \to 0.$$

Here $\mathcal{O}(-Y)$ is the dual bundle of $\mathcal{O}(Y)$. This means that sections $s \in \Gamma(\mathcal{O}(-Y))$ transform as $s_U = t_{UV} s_V$ with $t_{UV} = f_V / f_U$, where Y has defining function $f_U = 0$ over an open set U. The transformation relation shows that the combination $s_U f_U$ is a well-defined function on the manifold.

To explain the exact sequence, over an open set U where Y has defining function $f_U = 0$, a local holomorphic section $s_U \in \mathcal{O}(-Y)_U$ gets sent to $s_U f_U \in \mathcal{O}_U$ which is a holomorphic function on U vanishing along Y.

Tensoring with $\otimes\Omega_X^p$ implies the exact sequence

$$0 \to \Omega_X^p(-Y) \to \Omega_X^p \to \Omega_X^p|_Y \to 0.$$

The corresponding long exact sequence in cohomology gives

$$H^q(X, \Omega_X^p(-Y)) \to H^q(X, \Omega_X^p) \to H^q(Y, \Omega_X^p|_Y) \to H^{q+1}(X, \Omega_X^p(-Y))$$

We note that

$$\begin{aligned} \dim H^q(X, \Omega_X^p(-Y)) &= \dim H^{n-q}(X, (\Omega^p)^* \otimes \mathcal{O}(Y) \otimes K_X) \\ &= \dim H^{n-q}(\Omega^{n-p}(\mathcal{O}(Y)) \\ &= 0 \end{aligned} \tag{2.28}$$

by Serre duality and the vanishing theorem (Theorem 2.15) when $(n-q)+(n-p) > n$. Also considering this with q replaced by $q+1$, we see that when $p+q < n-1$ we have

$$0 \to H^q(X, \Omega_X^p) \to H^q(Y, \Omega_X^p|_Y) \to 0 \tag{2.29}$$

and so $H^q(X, \Omega_X^p) \to H^q(Y, \Omega_X^p|_Y)$ is an isomorphism.

Next, we will show that $H^q(Y, \Omega_X^p|_Y) \to H^q(Y, \Omega_Y^p)$ is an isomorphism. For this we use the exact sequence of vector spaces

$$0 \to (N|_Y)_y^* \otimes (\Lambda_Y^{p-1,0})_y \to (\Lambda_X^{p,0}|_Y)_y \to (\Lambda_Y^{p,0})_y \to 0. \tag{2.30}$$

Here $y \in Y$ and coordinates are chosen over an open set $U \subset X$ such that $U \cap \{z^n = 0\} = U \cap Y$, and

$$\begin{aligned} (N|_Y)_y^* &= \text{span}\,\{dz^n\} \\ (\Lambda_X^{1,0}|_Y)_y &= \text{span}\,\{dz^1, \ldots, dz^{n-1}, dz^n\} \\ (\Lambda_Y^{1,0})_y &= \text{span}\,\{dz^1, \ldots, dz^{n-1}\} \end{aligned} \tag{2.31}$$

and we may multiply the generators by local holomorphic functions on Y. The sequence (2.30) implies

$$0 \to \mathcal{O}(-Y)|_Y \otimes \Omega_Y^{p-1} \to \Omega_X^p|_Y \to \Omega_Y^p \to 0,$$

as sheaves over Y. This is because

$$(N|_Y)^* = \mathcal{O}(-Y)$$

which can be seen as follows: if there are two sets of coordinates $z, \tilde{z}$ where both $z^n = 0$ and $\tilde{z}^n = 0$ locally cut out Y, then $\tilde{z}^n(z) = z^n f(z)$, where $f(z)$ is the transition function for $\mathcal{O}(Y)$. Note that $f(z)$ is non-vanishing; this is because $Y = \{\tilde{z}^n(z) = 0\}$ smooth means that $\partial_{z^n}\tilde{z}^n(y) \neq 0$. Next, $d\tilde{z}^n = \frac{\partial \tilde{z}^n}{\partial z^i} dz^i$ implies $d\tilde{z}^n|_Y = f(z) dz^n|_Y$. This is the transformation law for the local frame dz^n, so components of $(N|_Y)^*$ transform by the inverse $1/f$ which is why the dual $\mathcal{O}(-Y)$ appears.

Thus

$$H^q(Y, \Omega_Y^{p-1}(-Y)) \to H^q(Y, \Omega_X^p|_Y) \to H^q(Y, \Omega_Y^p) \to H^{q+1}(Y, \Omega_Y^{p-1}(-Y)).$$

We apply Serre duality and the vanishing theorem as before to obtain

$$\dim H^{q+1}(Y, \Omega_Y^{p-1}(-Y)) = \dim H^{n-q-1}(Y, \Omega_Y^{n-p+1} \otimes \mathcal{O}(Y)) = 0.$$

Here we used that $\mathcal{O}(Y)|_Y \to Y$ is positive if $\mathcal{O}(Y) \to X$ is positive. The dimension of Y is $n-1$, so the vanishing theorem applies if

$$(n-q-1)+(n-p+1)>(n-1)$$

which holds for $p+q \leqslant n-2$. Therefore

$$0 \to H^q(Y, \Omega_X^p|_Y) \to H^q(Y, \Omega_Y^p) \to 0$$

so $H^q(Y, \Omega_X^p|_Y) \to H^q(Y, \Omega_Y^p)$ is an isomorphism, which combined with (2.29) gives that

$$H^q(X, \Omega_X^p) \to H^q(Y, \Omega_Y^p)$$

is an isomorphism.

$\square$

3 Deformations of Complex Manifolds

3.1 Families of complex manifolds

From a complex manifold given by

$$X = \{P = 0\} \subset \mathbb{P}^n,$$

we can consider a 1-parameter family X_t by inserting a parameter $t \in \mathbb{C}$ in front of one of the polynomial coefficients. A famous example [3] is

$$X_t = \left\{ \sum_{k=0} Z_k^5 - t \prod_{k=0}^{4} Z_k = 0 \right\} \subset \mathbb{P}^4.$$

The total space including the parameter t is

$$\mathcal{X} = \{(p, t) \in X_t \times \mathbb{C}\}.$$

We can also consider families where $t \in \mathbb{C}^r$, and we use the notation $\Delta \subset \mathbb{C}^r$ for a ball of radius 1. The formal definition of a family of complex manifolds is:

Definition 3.1. *Let X_0 be a compact complex manifold. A family of deformations of X_0 over $\Delta \subset \mathbb{C}^r$ is given by $\pi : \mathcal{X} \to \Delta$ where $\mathcal{X}$ is a complex manifold, π is a holomorphic map, $\pi^{-1}(0) = X_0$, and the Jacobian of π has maximal rank.*

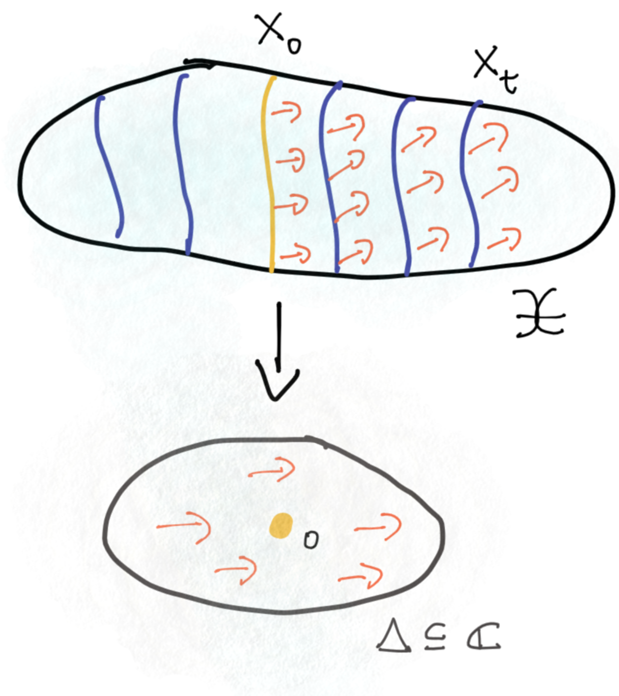
Using the definition $\pi : \mathcal{X} \to \Delta$ and the maximal rank theorem for holomorphic submersions, we may cover $\mathcal{X} = \bigcup_i U_i$ so that local coordinates on U_i are of the form $(z^1, \ldots, z^n, t)$, where z^i are holomorphic coordinates on $U_i \cap X_t$, and

$$\pi(z^1, \ldots, z^n, t) = t.$$

Change of coordinates on an overlap (z, t), $(\tilde{z}, \tilde{t})$ are of the form

$$\tilde{z}^k = f^k(z^1, \ldots, z^n, t), \quad \tilde{t} = t. \tag{3.1}$$

We will now show that in a family of complex manifolds, all underlying smooth manifolds are diffeomorphic.



Lemma 3.2. *Let $\pi : \mathcal{X} \to \Delta$ be a family of compact complex manifolds.*

Let $t_1 \in \Delta$. From any path $\gamma : [0,1] \to \Delta$ with $\gamma(0) = 0$ and $\gamma(1) = t_1$, we can construct a 1-parameter family of diffeomorphisms $\Theta_s : X_0 \to X_{\gamma(s)}$ such that $\Theta_0 = id$.

In particular X_{t_1} is diffeomorphic to X_0 for all $t_1 \in \Delta$.

Proof. Let $\gamma(s)$ be a path on Δ from 0 to t_0. Extend the vector field $\dot{\gamma}$ arbitrarily to all of Δ. Let $\{b^i\}$ be real coordinates on Δ, and write

$$\dot{\gamma}(b) = \dot{\gamma}^i(b)\frac{\partial}{\partial b^i}.$$

We will lift up the vector field on the base $\dot{\gamma} \in \Gamma(TB, B)$ using a partition of unity. Cover $\mathcal{X}$ with finitely many coordinates charts $(z^1, \ldots, z^n, b^1, \ldots, b^n)$ as described earlier where $\pi(z,b) = b$. Let ρ_α be a partition of unity ($\sum \rho_\alpha = 1$, $\mathrm{supp}(\rho_\alpha) \subset U_\alpha$) subordinate to this cover. Define on $\mathcal{X}$ the vector field

$$V = \sum_\alpha \rho_\alpha \left[\dot{\gamma}^i \frac{\partial}{\partial b^i_{U_\alpha}}\right].$$

It may be clearer to write V using fixed coordinates on say $U_0 \subset \mathcal{X}$, in which case

$$V = \dot{\gamma}^i \frac{\partial}{\partial b^i_{U_0}} + \sum_\alpha \rho_\alpha \dot{\gamma}^i \frac{\partial z^k_{U_0}}{\partial b^i_{U_\alpha}} \frac{\partial}{\partial z^k_{U_0}}.$$

So we get a lifted vector field $V \in \Gamma(T\mathcal{X})$ with $\pi_* V = \dot{\gamma}$. Next, we solve the ODE system on $\mathcal{X}$

$$\frac{d}{d\varepsilon}\Theta_\varepsilon = V, \quad \Theta_0 = id,$$

which is well-known to produce a 1-parameter family of diffeomorphisms $\Theta_\varepsilon : \mathcal{X} \to \mathcal{X}$ together with inverses $\Theta_{-\varepsilon} : \mathcal{X} \to \mathcal{X}$ satisfying $\frac{d}{d\varepsilon}\Theta_{-\varepsilon} = -V$ and $\Theta_\varepsilon \circ \Theta_{-\varepsilon} = id$.

The last thing to check is that this construction produces diffeomorphisms from fiber to fiber $\Theta_\varepsilon : X_0 \to X_{\gamma(\varepsilon)}$. Fix $x \in X_0$ and consider the function $f(\varepsilon) = \pi \circ \Theta_\varepsilon(x) - \gamma(\varepsilon)$. Then

$$f(0) = 0, \quad f'(\varepsilon) = \pi_* V - \dot{\gamma} = 0.$$

Therefore $f(\varepsilon) \equiv 0$ and so $\Theta_\varepsilon(x) \in X_{\gamma(\varepsilon)}$.

A similar argument shows $\Theta_{-\varepsilon} : X_{\gamma(\varepsilon)} \to X_0$: consider $x \in X_{\gamma(\varepsilon)}$, $f(r) = \pi \circ \Theta_{-r}(x) - \gamma(\varepsilon - r)$, and prove $f(\varepsilon) = 0$. Since $\Theta_\varepsilon \circ \Theta_{-\varepsilon} = id$, we have that $\Theta_\varepsilon : X_0 \to X_{\gamma(\varepsilon)}$ is a family of diffeomorphisms.

$\square$

Looking relative to the moving family of diffeomorphisms, we may regard X_t as the fixed differentiable manifold X_0, and let the complex structure tensor J_t vary in t. More precisely: let $\mathcal{X} \to \Delta$ be a family of complex manifolds, so that $(X_t, \check{J}_t)$ is a complex manifold for each parameter $t \in \Delta$. Take a path $\gamma : (-\varepsilon, \varepsilon) \to \Delta$ with $\gamma(0) = 0$, $\dot{\gamma}(0) \neq 0$, and so from Lemma 3.2 we obtain a family of diffeomorphisms $\Theta_t : X_0 \to X_{\gamma(t)}$. Then we may consider

$$(X_0, J_t), \quad J_t = (\Theta_t)_*^{-1} \check{J}_{\gamma(t)} (\Theta_t)_*$$

which is a 1-parameter family of complex structures on a fixed differentiable manifold X_0. This defines new complex structures because $J_t^2 = -id$ and $N(J_t) = 0$ (as because N transforms as a tensor under coordinate transformation). We now define the Nijenhuis tensor N.

On a differentiable manifold X, an almost-complex structure is a tensor $J \in \Gamma(\mathrm{End} T_{\mathbb{R}} X)$ satisfying $J^2 = -id$. The Newlander-Nirenberg theorem (see e.g. [7] for a proof) states that an almost complex structure J comes from holomorphic coordinates $\{z^\alpha\}$ with $J\frac{\partial}{\partial z^\alpha} = i\frac{\partial}{\partial z^\alpha}$, $J\frac{\partial}{\partial \bar{z}^\alpha} = -i\frac{\partial}{\partial \bar{z}^\alpha}$ if and only if the Nijenhuis tensor

$$N^k{}_{ij} = \frac{1}{4}\bigg(J^r{}_i \partial_r J^k{}_j + J^k{}_r \partial_j J^r{}_i - (i \leftrightarrow j) \bigg). \tag{3.2}$$

vanishes identically: $N^k{}_{ij} = 0$. It can also be checked that the components of $N^k{}_{ij}$ transform correctly so that N is a legitimate tensor.

Remark 3.3. Here is some motivation regarding N. From an almost complex-structure J on a smooth manifold X, we may split $T_{\mathbb{C}} X = T^{1,0}X \oplus T^{0,1}X$, where $T^{1,0}X$ is the $+i$ eigenspace of J and $T^{0,1}X$ is the $-i$ eigenspace of J. The Nijenhuis tensor $N = \frac{1}{2} N^p{}_{mn}\, dx^m \wedge dx^n \otimes \partial_p$ can be shown to satisfy $N(U, V) = -[U, V]^{0,1}$ for $U, V \in \Gamma(T^{1,0}X)$. In other words, $N = 0$ if and only if $[U, V] \in T^{1,0}X$ for all $U, V \in T^{1,0}X$. Hence N measures the failure of the subbundle $T^{1,0}X \subset T_{\mathbb{C}} X$ being closed under the Lie bracket.

Let $\mathcal{X} \to \Delta$ be a family of complex manifolds with central fiber (X, J), together with a path $\gamma : (-\varepsilon, \varepsilon) \to \Delta$ with $\gamma(0) = 0$ and $\dot{\gamma}(0) \neq 0$. From this data, we would like to produce an element

$$[\eta] \in H^1(X, T^{1,0}X).$$

Given our discussion so far, from this information we can create a path of complex structures $J_t \in \Gamma(\mathrm{End}TX)$ on the fixed differentiable manifold X with $J_0 = J$ satisfying the constraints $J_t^2 = -id$ and $N(J_t) = 0$. We will now show:

- Differentiation $\eta = \dot{J}(0)$ produces an element $\eta \in \Lambda^{0,1}(T^{1,0}X)$ which satisfies $\bar{\partial}\eta = 0$ and hence defines a Dolbeault cohomology class $[\eta] \in H^1(X, T^{1,0}X)$.

- Different choices of diffeomorphisms $\Theta_t : X \to X_{\gamma(t)}$ in Lemma 3.2 produce the same class $[\eta] \in H^1(X, T^{1,0}X)$.

We start by differentiating $J_t^2 = -id$. Take $\frac{d}{dt}|_{t=0}$ and obtain

$$\dot{J}^i{}_k J^k{}_j + J^i{}_k \dot{J}^k{}_j = 0. \tag{3.3}$$

Here $\dot{J} = \frac{d}{dt}|_{t=0} J_t$. We will work in coordinates $x = (z^1, \dots, z^n, \bar{z}^1, \dots, \bar{z}^n)$ on the fixed complex manifold (X, J) where $\{z^\alpha\}$ are holomorphic coordinates on (X, J). Let α, β denote holomorphic coordinates, so that $\alpha \in \{1, \dots, n\}$ and $\partial_\alpha = \frac{\partial}{\partial z^\alpha}$, $\partial_{\bar{\alpha}} = \frac{\partial}{\partial \bar{z}^\alpha}$. Let i, j, p denote indices for the real coordinates x, so that $i \in \{1, \dots, n, \bar{1}, \dots, \bar{n}\}$ and so we could have $i = \alpha$ or $i = \bar{\alpha}$. Summations over i, j, k run over both unbarred coordinates z^α and barred coordinates $\bar{z}^\alpha$.

With this convention, then $J^\alpha{}_\beta = i\delta^\alpha{}_\beta$, $J^{\bar{\alpha}}{}_{\bar{\beta}} = -i\delta^\alpha{}_\beta$ and $J^\alpha{}_{\bar{\beta}} = J^{\bar{\alpha}}{}_\beta = 0$. Then (3.3) with $i = \alpha$, $j = \beta$ implies

$$\dot{J}^\alpha{}_\beta = 0.$$

Similarly $\dot{J}^{\bar{\alpha}}{}_{\bar{\beta}} = 0$. Since $J(t)$ is real, so is $\dot{J}$, and hence $\dot{J}$ is determined by the components

$$\eta^\alpha{}_{\bar{\beta}} := \dot{J}^\alpha{}_{\bar{\beta}}.$$

Next, we differentiate $N(J_t) = 0$. Taking $\frac{d}{dt}|_{t=0}$ of (3.2) gives

$$0 = \dot{J}^r{}_{\bar{\beta}} \partial_r J^\alpha{}_{\bar{\gamma}} + J^r{}_{\bar{\beta}} \partial_r \dot{J}^\alpha{}_{\bar{\gamma}} + \dot{J}^\alpha{}_r \partial_{\bar{\gamma}} J^r{}_{\bar{\beta}} + J^\alpha{}_r \partial_{\bar{\gamma}} \dot{J}^r{}_{\bar{\beta}} - (\bar{\beta} \leftrightarrow \bar{\gamma})$$

which becomes

$$0 = 2\Big[-i\partial_{\bar{\beta}} \dot{J}^\alpha{}_{\bar{\gamma}} + i\partial_{\bar{\gamma}} \dot{J}^\alpha{}_{\bar{\beta}} \Big].$$

Since $\eta^\alpha{}_{\bar{\beta}} \in \Gamma(T^{1,0}X \otimes \Lambda^{0,1})$, this implies

$$\bar{\partial}\eta = 0, \quad [\eta] \in H^1(X, T^{1,0}X)$$

and we call $[\eta]$ the Kodaira-Spencer class.

We now want to show that $[\eta] \in H^1(X, T^{1,0}X)$ is independent of the choice of family of diffeomorphisms in Lemma 3.2. Suppose from the path $\gamma(t)$ on Δ, we produce two families of diffeomorphisms

$$\Theta_t : X \to X_{\gamma(t)}, \quad \Psi_t : X \to X_{\gamma(t)}, \quad \Theta_0 = \Psi_0 = id.$$

From the complex manifold $(X_t, \check{J}_t)$, we produce a family of complex structures on the fixed X_0 as before by $J_t = (\Theta_t)_*^{-1} \check{J}_{\gamma(t)} (\Theta_t)_*$ and $\tilde{J}_t = (\Psi_t)_*^{-1} \check{J}_{\gamma(t)} (\Psi_t)_*$. These are related by

$$\tilde{J}_t = (f_t)_* J_t (f_t)_*^{-1}$$

with $f_t = \Psi_t^{-1} \circ \Theta_t$. Then f_t be a 1-parameter family of diffeomorphisms with $\frac{d}{dt}|_{t=0} f_t = V$ (for some vector field V) and $f_0 = id$.

We now compute $\frac{d}{dt}|_{t=0}$ of $\tilde{J}_t$. Let $y^\alpha = f_t^\alpha(x)$ be a change of coordinates by the diffeomorphism. The formulas for the pushfoward $f_* : T_p M \to T_{f(p)} M$ are

$$(f_* V)^a (f(p)) = \frac{\partial y^a}{\partial x^i}(p) V^i(p), \quad (f_*^{-1} V)^a(p) = \frac{\partial x^i}{\partial y^a}(f(p)) V^a(f(p)),$$

and so acting J_t on $V \in T_{f(p)} M$ gives

$$[\tilde{J}_t{}^a{}_b V^b](f(p)) = \frac{\partial y^a}{\partial x^k}(p) J_t(p)^k{}_i \frac{\partial x^i}{\partial y^a}(f(p)) V^a(f(p))$$

The components of J_t are then

$$\tilde{J}_t^a{}_b(y_t(x)) = \frac{\partial y_t^a}{\partial x^k}(x) J_t(x)^k{}_i \frac{\partial x^i}{\partial y_t^b}(y_t(x)).$$

Differentiating this in time at $t = 0$ along a path with $y_0(x) = x$ and $\dot{y}(0) = V$ gives

$$\dot{\tilde{J}}^a{}_b + \partial_k J^a{}_b V^k = \partial_k V^a J^k{}_b + \dot{J}^a{}_b + J^a{}_i \frac{d}{dt}\bigg|_{t=0} \frac{\partial x^i}{\partial y^b}(y_t(x))$$

In complex coordinates, the second term on the left is zero since the components $J^a{}_b$ are constant. For the last term, we differentiate in time the chain rule identity

$$\frac{\partial x^i}{\partial y^k}(y(x)) \frac{\partial y^k}{\partial x^j}(x) = \frac{\partial}{\partial x^j} x^i = \delta^i{}_j$$

to obtain

$$\frac{d}{dt}\bigg|_{t=0} \frac{\partial x^i}{\partial y^j}(y_t(x)) = -\partial_j V^i.$$

Therefore

$$\dot{\tilde{J}}^a{}_b = \dot{J}^a{}_b + \partial_k V^a J^k{}_b - J^a{}_i \partial_b V^i.$$

We showed earlier that considering $J^2 = -id$ implies that the non-zero contributions are $\eta^\alpha{}_{\bar{\beta}} = \dot{J}(0)^\alpha{}_{\bar{\beta}}$ and its conjugate. So we let $a = \alpha$, $b = \bar{\beta}$ and obtain

$$\tilde{\eta}^\alpha{}_{\bar{\beta}} = \eta^\alpha{}_{\bar{\beta}} - 2i \partial_{\bar{\beta}} V^\alpha.$$

It follows that

$$\tilde{\eta} = \eta - 2i \bar{\partial} V^{1,0}$$

and so $[\eta] = [\tilde{\eta}] \in H^1(T^{1,0} X)$.

Remark 3.4. Let (X, J) be a complex manifold. This calculation shows that deformations $\tilde{J}_t = (f_t)_* J (f_t)_*^{-1}$ created by 1-parameter families of diffeomorphisms f_t produce the zero class $[\eta] = 0 \in H^1(X, TX)$ (since $\dot{J} = 0$). Deformations of complex structure coming from families of diffeomorphisms are not counted by $[\eta] \in H^1(X, TX)$.

A central question in deformation theory is the inverse problem: given $\eta \in H^1(T^{1,0}X)$, does it come from a family $(X(t), J(t))$ of complex manifolds with $\dot{J}(0) = \eta$? This statement is not true in general, but it is true for Kähler Calabi-Yau manifolds (this is the Bogomolov-Tian-Todorov theorem). For more references on this topic, see for example [20, 16, 14].

3.2 Semi-continuity theorem

We start with an illustrative example from linear algebra. Let $A_t : \mathbb{R}^n \to \mathbb{R}^n$ be a family of symmetric real $n \times n$ matrices with entries continuously varying in $t \in \mathbb{R}$. Then there exists $\varepsilon > 0$ such that for all $|t| < \varepsilon$, then

$$\dim \ker A_t \leqslant \dim \ker A_0.$$

The dimension of $\ker A_t$ may jump at $t = 0$, as seen from e.g.

$$A_t = \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}.$$

Returning to complex geometry, let (X, J_t) be a family of complex structures on a compact manifold X with $J_0 = J$. A differential form $\alpha \in \Lambda^k(X)$ has different decompositions into (p, q) types for each parameter t. Write $\Lambda_0^{1,0}$ for $(1, 0)$ forms with respect to the initial structure J, and $\Lambda_t^{1,0}$ for $(1, 0)$ forms with respect to J_t. We can decompose $\alpha \in \Lambda^1(X)$ into (p, q) components via the formula

$$\begin{aligned} \alpha &= \frac{1}{2}(\alpha - iJ_t\alpha) + \frac{1}{2}(\alpha + iJ_t\alpha) \\ &:= (\alpha)_t^{1,0} + (\alpha)_t^{0,1} \end{aligned} \tag{3.4}$$

Here we define $J\alpha(X) = \alpha(JX)$. With this notation, we have that $\alpha \in \Lambda_0^{1,0}$ if and only if $J\alpha = +i\alpha$. The map $\varphi_t : \Lambda_0^{1,0} \to \Lambda_t^{1,0}$ given by

$$\alpha \mapsto (\alpha)_t^{1,0}$$

is an isomorphism for small t, since $\varphi_0 = id$ and φ_t varies continuously as seen from the explicit expression above. Similarly, we obtain isomorphisms

$$\varphi_t : \Lambda_0^{p,q} \to \Lambda_t^{p,q}$$

for small t. Recall that

$$h^{p,q}(X_t) = \dim \ker\{\Delta_{\bar{\partial}_t} : \Lambda_t^{p,q} \to \Lambda_t^{p,q}\}.$$

We will show that $h^{p,q}(X_t)$ is an upper-semicontinuous function, meaning

$$h^{p,q}(X_t) \leqslant h^{p,q}(X_0), \quad |t| < \varepsilon.$$

To view all operators on the same space, instead of $\Delta_{\bar{\partial}}$ we will use

$$L_t : \Lambda_0^{p,q} \to \Lambda_0^{p,q}, \quad L_t = \varphi_t^{-1} \circ \Delta_{\bar{\partial}_t} \circ \varphi_t,$$

and show

$$\dim\ker L_t \leqslant \dim\ker L_0, \quad |t| < \varepsilon.$$

To prove this, we will need some PDE estimates.

Let $E \to X$ be a vector bundle over a compact manifold. Cover X by finitely many trivializations $X = \bigcup_{i=1}^N U_i$ where $B_i \subset U_i \subset \mathbb{C}^n$ are balls of radius 1 still covering X. Let $\psi \in \Gamma(X, E)$ and let ψ_{U_i} denote the vector valued function of the components of ψ in the trivialization U_i. Let $0 < \alpha < 1$. Define

$$\|\psi\|_{C^{k,\alpha}} := \sup_i \|\psi_{U_i}\|_{C^{k,\alpha}(B_i)}$$

where for a function $f : B \to \mathbb{R}^p$ the Hölder norm is

$$\|f\|_{C^{k,\alpha}(B)} = \|f\|_{C^k(B)} + \sup_{|I|=k} \sup_{x \neq y} \frac{|D^I f(x) - D^I f(y)|}{|x-y|^\alpha},$$

where $\|f\|_{C^k(B)} = \sup_{|I|=k} \sup_B |D^I f|$. We write $\psi \in C^{k,\alpha}(X, E)$ if ψ is a k-times differentiable section of E with finite $\|\cdot\|_{C^{k,\alpha}}$ norm. The main features of $C^{k,\alpha}$ spaces for our purposes are:

- $C^{k,\alpha}(X, E)$ is a Banach space.
- Compactness: suppose $\{\psi_n\} \in C^{k,\alpha}(X, E)$ is a sequence of sections such that

$$\|\psi_n\|_{C^{k,\alpha}} \leqslant C$$

for uniform constant $C > 0$. Let $0 < \alpha' < \alpha$. Then there exists a limiting section $\psi_\infty \in C^{k,\alpha}$ and a subsequence $\{\psi_{n_k}\}$ such that $\psi_{n_k} \to \psi_\infty$ in $C^{k,\alpha'}$.

- The Schauder estimates. (Theorem 3.5 below)

Let us prove that $C^{0,\alpha}(X, E) := C^\alpha$ is a Banach space and its compactness property. It is routine to check that $\|\cdot\|_{C^\alpha}$ is a norm. To show completeness, we must show that if $\{\psi_n\}$ is a Cauchy sequence, then ψ_n converges to $\psi_\infty \in C^\alpha$. By Arzela-Ascoli applied on each coordinate ball B_i, a subsequence ψ_{n_k} converges in C^0 to a continuous limit ψ_∞, and since $\{\psi_n\}$ is Cauchy then the full sequence converges

$$\psi_n \to \psi_\infty \text{ in } C^0.$$

The limit section ψ_∞ is in C^α, since for $x, y \in B_i$ with $|x - y| = \delta$, then

$$\begin{aligned} \frac{|\psi_\infty(x) - \psi_\infty(y)|}{|x-y|^\alpha} &\leqslant \frac{|\psi_\infty(x) - \psi_k(x)|}{|x-y|^\alpha} + \frac{|\psi_k(x) - \psi_k(y)|}{|x-y|^\alpha} + \frac{|\psi_k(y) - \psi_\infty(y)|}{|x-y|^\alpha} \\ &\leqslant C \end{aligned} \tag{3.5}$$

for k large enough such that $\|\psi_\infty - \psi_k\|_{C^0} < \delta$. To show the sequence converges in C^α, let $\varepsilon > 0$. There exists N such that if $k, \ell \geqslant N$ then for all $x \neq y$ there holds

$$\frac{|[\psi_k - \psi_\ell](x) - [\psi_k - \psi_\ell](y)|}{|x-y|^\alpha} \leqslant \varepsilon.$$

Fix x, y and send $\ell \to \infty$ to conclude

$$\|\psi_k - \psi_\infty\|_{C^\alpha} < \varepsilon.$$

We now show compactness. Suppose $\{\psi_n\} \in C^\alpha$ is a sequence with $\|\psi_n\|_{C^\alpha} \leqslant C$. By the Arzela-Ascoli theorem applied to coordinate balls B_i, we obtain a subsequence ψ_{n_k} converging in C^0 to a continuous limit ψ_∞. The limiting section is Hölder continuous $\psi_\infty \in C^\alpha$ by estimate (3.5). To see convergence in $0 < \alpha' < \alpha$, we let $v_k = \psi_{n_k} - \psi_\infty$ and write

$$\frac{|v_k(x) - v_k(y)|}{|x-y|^{\alpha'}} = \left(\frac{|v_k(x) - v_k(y)|}{|x-y|^{\alpha}} |v_k(x) - v_k(y)|^{(\alpha/\alpha')-1} \right)^{\alpha'/\alpha}.$$

This goes to zero as $k \to \infty$.

Next, we state the Schauder estimates.

Theorem 3.5. *(Schauder estimates I) Let $E, F \to X$ be vector bundles over a compact manifold. Let $L : \Gamma(X,E) \to \Gamma(X,F)$ be an elliptic operator of order k. There exists $C > 0$ such that for all sections $s \in C^{k,\alpha}(X,E)$, then*

$$\|s\|_{C^{k,\alpha}(X)} \leqslant C(\|s\|_{L^\infty} + \|Ls\|_{C^{0,\alpha}}).$$

Here C only depends on the constant of ellipticity and the C^α norms of the coefficients of L.

For a reference, see [19], remark after Theorem 4.3 in the appendix.

We can upgrade this estimate for sections $s \in (\ker L)^\perp$.

Theorem 3.6. *(Schauder estimates II) Let $E, F \to X$ be complex vector bundles over a compact complex manifold. Let $L : \Gamma(X,E) \to \Gamma(X,F)$ be an elliptic operator of order k. Let H be a metric on E and ω a hermitian metric on X. There exists $C > 0$ such that for all $s \in C^{k,\alpha}(X,E)$ with*

$$s \in (\ker L)^\perp$$

then

$$\|s\|_{C^{k,\alpha}(X)} \leqslant C\|Ls\|_{C^{0,\alpha}}.$$

Here $s \in (\ker L)^\perp$ means: $(s,\varphi)_{L^2} = 0$ for all $\varphi \in \Gamma(X,E)$ with $L\varphi = 0$, and the L^2 inner product on $\Gamma(X,E)$ is as before: $(s,\varphi)_{L^2} = \int_X \langle s, \varphi\rangle_H \, \omega^n$.

Proof. Suppose this estimate is false. Then there exists a sequence $\{s_i\} \in (\ker L)^\perp \cap C^{k,\alpha}$ such that

$$\|s_n\|_{C^{k,\alpha}} \geqslant M_n \|Ls_n\|_{C^{0,\alpha}}, \quad M_n \to \infty.$$

Let $u_n = s_n / \|s_n\|_{C^{k,\alpha}}$. Then $\{u_n\} \in (\ker L)^\perp$ satisfies

$$\|Lu_n\|_{C^{0,\alpha}(X)} \leqslant \frac{1}{M_n}, \quad \|u_n\|_{C^{k,\alpha}} = 1.$$

From $\|u\|_{C^{k,\alpha}} = 1$, compactness of Hölder space allows us to extract a subsequence $u_n \to u_\infty$ converging in the $C^{k,\alpha'}$ norm with $0 < \alpha' < \alpha$. Since $\|Lu_n\|_{C^{0,\alpha'}} \to 0$, the limit satisfies $Lu_\infty = 0$.

Next, $u_n \in (\ker L)^\perp$ and $u_\infty \in \ker L$ implies $(u_n, u_\infty)_{L^2} = 0$ for all n, and we conclude

$$u_\infty = 0.$$

We want to obtain a contradiction, but we cannot take a limit of $\|u_n\|_{C^{k,\alpha}} = 1$ since $\alpha > \alpha'$. However, by the usual Schauder estimates, for $n \gg 1$ large enough we have

$$1 = \|u_n\|_{C^{k,\alpha}} \leqslant C(\|u_n\|_{L^\infty} + \|Lu_n\|_{C^{0,\alpha}}) \leqslant C\|u_n\|_{L^\infty} + \frac{1}{2}$$

and so $\|u_n\|_{L^\infty} \geqslant 1/2C$ for all n large. Taking a limit gives $\|u_\infty\|_{L^\infty} > 0$ which is a contradiction.

□

From the Schauder estimates, we can deduce the semi-continuity theorem.

Theorem 3.7. *Let $E, F \to X$ be complex vector bundles over a compact manifold. Let $L_t : \Gamma(X,E) \to \Gamma(X,F)$ be a continuous family of elliptic operators of order k. Then there exists $\varepsilon > 0$ so that*

$$\dim\ker L_t \leqslant \dim\ker L_0, \quad |t| < \varepsilon$$

so that $\dim\ker L_t$ *is upper semi-continuous.*

Proof. Let $u \in \Gamma(X,E)$ satisfy $u \in (\ker L_0)^\perp$. Then by the Schauder estimate,

$$\begin{aligned} \|u\|_{C^{k,\alpha}} &\leqslant C\|L_0 u\|_{C^\alpha} \\ &\leqslant C\|L_t u\|_{C^\alpha} + C\|(L_t - L_0)u\|_{C^\alpha} \\ &\leqslant C\|L_t u\|_{C^\alpha} + C\varepsilon\|u\|_{C^{k,\alpha}} \end{aligned} \tag{3.6}$$

if $\|L_t - L_0\|_{C^\alpha} \leqslant \varepsilon$. For $\varepsilon > 0$ small enough, then $C\varepsilon < \frac{1}{2}$ and we obtain

$$\|u\|_{C^{k,\alpha}} \leqslant 2C\|L_t u\|_{C^\alpha}.$$

It follows that

$$(\ker L_0)^\perp \cap (\ker L_t) = \{0\}.$$

From

$$(\ker L_t) \subset \Gamma(X,E) = (\ker L_0) \oplus (\ker L_0)^\perp,$$

we have $\ker L_t \subseteq \ker L_0$. □

To conclude semi-continuity of Hodge numbers, we let $L_t = \varphi_t^{-1} \circ \Delta_{\bar\partial} \circ \varphi_t : \Lambda_0^{p,q} \to \Lambda_0^{p,q}$ as before. Then

$$h^{p,q}(X_t) \leqslant h^{p,q}(X_0), \quad |t| < \varepsilon$$

and we see that $t \mapsto \dim H^q(X_t, \Omega^p)$ is upper semi-continuous in a family of complex manifolds.

3.3 Stability of Kähler metrics

Kodaira-Spencer's stability theorem states that if X_0 is a compact Kähler manifold, then any deformation X_t admits a Kähler metric for small enough t [21].

After this section, we will be restricting our attention from general complex manifolds to simply connected Calabi-Yau threefolds. We will in this section give the proof of Kodaira-Spencer's stability theorem assuming that X_0 is a simply connected Calabi-Yau threefold. Explicitly, the only extra hypothesis that we will use is $H^{0,2}(X_0) = 0$, so the proof here applies to any complex manifold X_0 satisfying this property. Our proof under this hypothesis is simpler compared compared to Kodaira-Spencer [21], and we will follow the argument and exposition in Fu-Li-Yau [11]. The setup in Fu-Li-Yau [11] is different as it concerns balanced metrics rather than Kähler metrics and the central fiber X_0 has nodal singularities, but the outline of the argument is readily adapted.

Let ω_0 be the Kähler metric on X_0, and Θ_t be a family of diffeomorphisms $\Theta_t : X_0 \to X_t$. Then

$$\Theta_t^*\omega_0 = (\Theta_t^*\omega_0)^{2,0} + (\Theta_t^*\omega_0)^{1,1} + (\Theta_t^*\omega_0)^{0,2}$$

is a closed 2-form on X_t. To obtain a Kähler metric on X_t, it must be of type $(1,1)$, so we let

$$\chi_t = (\Theta_t^*\omega_0)^{1,1}.$$

But this is no longer closed.

$$\begin{aligned} 0 &= d(\Theta_t^*\omega_0) \\ &= \left[\partial(\Theta_t^*\omega_0)^{2,0}\right] + \left[\bar\partial(\Theta_t^*\omega_0)^{2,0} + \partial(\Theta_t^*\omega_0)^{1,1}\right] + \left[\bar\partial(\Theta_t^*\omega_0)^{1,1} + \partial(\Theta_t^*\omega_0)^{0,2}\right] + \left[\bar\partial(\Theta_t^*\omega_0)^{0,2}\right] \end{aligned}$$

By type considerations, we have

$$\bar\partial\chi_t = -\partial(\Theta_t^*\omega_0)^{0,2}, \quad \bar\partial(\Theta_t^*\omega_0)^{0,2} = 0. \tag{3.7}$$

So $\bar\partial\chi_t \to 0$ uniformly in all C^k norms. The claim is that we can correct χ_t by

$$\omega_\gamma = \chi_t + \partial\left[\bar\partial^\dagger\partial^\dagger\gamma\right] + \bar\partial\left[\partial^\dagger\bar\partial^\dagger\bar\gamma\right] \tag{3.8}$$

where $\gamma \in \Lambda^{1,2}$ solves

$$\bar\partial\partial\bar\partial^\dagger\partial^\dagger\gamma = -\bar\partial\chi_t. \tag{3.9}$$

This will produce a real $(1,1)$ form $\omega_\gamma \in \Lambda^{1,1}(X,\mathbb{R})$ satisfying

$$\bar\partial\omega_\gamma = \bar\partial\chi_t + \bar\partial\partial\bar\partial^\dagger\partial^\dagger\gamma = 0.$$

Taking the conjugate, we conclude $d\omega_\gamma = 0$. To show that ω_γ is a Kähler metric, we will show its positivity $\omega_\gamma > 0$. For this, we will show that the correction γ is small.

Roughly speaking, the strategy is to correct $\tilde\omega_t = \chi_t + \partial\alpha + \bar\partial\bar\alpha$ so that α solves $\bar\partial\partial\alpha = -\bar\partial\tilde\omega_t$ and so $\bar\partial\tilde\omega_t = 0$. If $\bar\partial\partial$ were an elliptic operator, one could try to use elliptic PDE theory to estimate $\|\alpha\| \leqslant C\|\partial\bar\partial\alpha\| = C\|\bar\partial\chi_t\|$ and show smallness of α since $\bar\partial\chi_t \to 0$ as $t \to 0$. To make this strategy work, we do not use α but rather γ with $\alpha = \bar\partial^\dagger\partial^\dagger\gamma$. Furthermore (3.9) is not quite an elliptic PDE for γ, but this equation can be modified to make this strategy work.

To make (3.9) elliptic, we consider the Kodaira-Spencer operator

$$E_t = \partial\bar{\partial}\bar{\partial}^\dagger\partial^\dagger + \bar{\partial}^\dagger\partial^\dagger\partial\bar{\partial} + \bar{\partial}^\dagger\partial\partial^\dagger\bar{\partial} + \partial^\dagger\bar{\partial}\bar{\partial}^\dagger\partial + \bar{\partial}^\dagger\bar{\partial} + \partial^\dagger\partial$$

and solve

$$E_t(\gamma_t) = \bar{\partial}\chi_t.$$

In the definition of E_t, the $\partial, \bar{\partial}$ are with respect to the complex structure at t, and the adjoints $\partial^\dagger, \bar{\partial}^\dagger$ are with respect to the non-Kähler metrics χ_t. The operator E_t is a 4th order elliptic operator as proved by Kodaira-Spencer. The first term in E_t matches with (3.9), so we will need to look for solutions in a space where all the other terms vanish; we see that if we can find a solution to $E(\gamma) = \bar{\partial}\omega$ with $d\gamma = 0$, we will solve (3.9).

By the Fredholm alternative, we can find a solution $\gamma_t \in (\ker E_t)^\perp$ if $\bar{\partial}\chi_t \perp \ker E_t$. Note that considering $(E_t\varphi, \varphi)$ and integrating by parts shows that

$$\ker E_t = \{\varphi : d\varphi = 0, \quad \bar{\partial}^\dagger\partial^\dagger\varphi = 0\}.$$

We claim that there exists $\mu \in \Lambda^{0,1}$ with

$$\bar{\partial}\chi_t = i\partial\bar{\partial}\mu_t. \tag{3.10}$$

Here we use the assumption of vanishing cohomology $H^{0,2}(X_0) = 0$. By the semi-continuity theorem,

$$h^{0,2}(X_t) \leqslant h^{0,2}(X_0) = 0$$

and by (3.7), we have $[(\Theta_t^*\omega_0)^{0,2}] \in H^{0,2}(X_t) = 0$ and so we can write $(\Theta_t^*\omega_0)^{0,2} = \bar{\partial}\nu$ for $\nu \in \Lambda^{0,1}$. By (3.7), we have $\bar{\partial}\chi_t = -\partial\bar{\partial}\nu$.

It follows from (3.10) that for $\beta \in \ker E_t$ then

$$\langle \bar{\partial}\chi_t, \beta\rangle = \langle \mu_t, \bar{\partial}^\dagger\partial^\dagger\beta\rangle = 0.$$

Therefore, we can find a solution to $E_t(\gamma_t) = \bar{\partial}\chi_t = i\partial\bar{\partial}\mu_t$. Next, we prove that such a solution is closed.

Lemma 3.8. *If $E(\gamma) = i\partial\bar{\partial}\mu$, then $d\gamma = 0$.*

Proof. We compute

$$\begin{aligned} 0 &= E_t(\gamma) - i\partial\bar{\partial}\mu_t \\ &= \partial\bar{\partial}\Big[\bar{\partial}^\dagger\partial^\dagger\gamma_t - i\mu_t\Big] + \partial^\dagger\Big[\bar{\partial}\bar{\partial}^\dagger\partial\gamma_t + \partial\gamma_t\Big] + \bar{\partial}^\dagger\Big[\partial^\dagger\partial\bar{\partial}\gamma_t + \partial\partial^\dagger\bar{\partial}\gamma_t + \bar{\partial}\gamma_t\Big] \\ &:= \partial\bar{\partial}\alpha_1 + \partial^\dagger\alpha_2 + \bar{\partial}^\dagger\alpha_3. \end{aligned} \tag{3.11}$$

Let $\sigma = \partial^\dagger\alpha_2 + \bar{\partial}^\dagger\alpha_3$. Then the equation above implies $d\sigma = 0$, and

$$\|\sigma\|^2 = (\alpha_2 + \alpha_3, (\partial + \bar{\partial})\sigma) = 0.$$

Setting $\sigma = 0$, we obtain

$$0 = \partial^\dagger\Big[\bar{\partial}\bar{\partial}^\dagger\partial\gamma_t + \partial\gamma_t\Big] + \bar{\partial}^\dagger\Big[\partial^\dagger\partial\bar{\partial}\gamma_t + \partial\partial^\dagger\bar{\partial}\gamma_t + \bar{\partial}\gamma_t\Big].$$

and taking an inner product with γ_t implies

$$0 = \|\bar{\partial}^\dagger \partial \gamma\|^2 + \|\partial \gamma\|^2 + \|\partial \bar{\partial} \gamma_t\|^2 + \|\partial^\dagger \bar{\partial} \gamma|^2 + \|\bar{\partial} \gamma\|^2$$

and so $\partial \gamma = 0$ and $\bar{\partial}\gamma = 0$. ☐

In summary, by the Fredholm alternative we have $\gamma_t \in (\ker E_t)^\perp$ with $E_t \gamma_t = \bar{\partial} \chi_t$ and $d\gamma_t = 0$. Therefore (3.9) holds and the corrected ω_γ (3.8) satisfies $d\omega_\gamma = 0$. To show ω_γ is a Kähler metric, we prove the estimate

$$\|\gamma_t\|_{C^3(X, g_0)} \leqslant C|t|. \tag{3.12}$$

Then since $\chi_t > (1/2)\omega_0$ for small t, we see that $\omega_\gamma > 0$ for small t.

Lemma 3.9. *There exists $C > 1$ independent of t such that for all t small, we can estimate*

$$\|\gamma_t\|_{C^{4,\alpha}} \leqslant C \|E_t \gamma_t\|_{C^\alpha}, \tag{3.13}$$

for all $\gamma_t \in (\ker E_t)^\perp$ with $d\gamma_t = 0$.

Proof. Suppose by contradiction that the estimate fails, and there is a sequence $t_i \to 0$ such that

$$\|\gamma_t\|_{C^{4,\alpha}} \geqslant C_i \|E_t \gamma_t\|_{C^\alpha},$$

with $C_i \to \infty$. Replacing γ_t by $\gamma_t / \|\gamma_t\|_{C^{4,\alpha}}$, so that we have a sequence γ_{t_i} with

$$\|\gamma_{t_i}\|_{C^{4,\alpha}} = 1, \quad \|E_{t_i} \gamma_{t_i}\|_{C^\alpha} \to 0.$$

The $E_{t_i} \to E_0$ smoothly, and after relabeling a subsequence we have that $\gamma_{t_i} \to \gamma_0$ in $C^{4,\alpha/2}$ with $E_0 \gamma_0 = 0$ and $d\gamma_0 = 0$. This limit is non-trivial due to the Schauder estimates

$$\|\gamma_t\|_{C^{4,\alpha}} \leqslant C(\|\gamma_t\|_{L^\infty} + \|E_t \gamma_t\|_{C^\infty})$$

where $C > 1$ is uniform in t as E_t is a smoothly varying family of elliptic operators on a fixed smooth manifold, and by compactness of t, the coefficients of E_t and its ellipticity are uniformly bounded. Taking $t \to \infty$ implies $\|\gamma_0\|_{L^\infty} \geqslant C^{-1}$.

Since $E_0 \gamma_0 = 0$, we showed earlier that $\bar{\partial}^\dagger \partial^\dagger \gamma_0 = 0$. Since $\bar{\partial} \gamma_0 = 0$, we have $\bar{\partial} \partial^\dagger \gamma_0 = 0$ by the Kähler identities on the Kähler central fiber X_0. Now $\partial^\dagger \gamma_0 \in H^{0,2}(X_0)$ and we are assuming $H^{0,2}(X_0) = 0$, so $\partial^\dagger \gamma_0 = \bar{\partial} q$. Therefore $\bar{\partial}^\dagger \bar{\partial} q = 0$, and so $\bar{\partial} q = 0$. It follows that

$$\partial \gamma_0 = 0, \quad \partial^\dagger \gamma_0 = 0.$$

Hence $\gamma \in \ker \Delta_\partial$, and in Kähler geometry $\Delta_\partial = \Delta_d$, so

$$d\gamma_0 = 0, \quad d^\dagger \gamma_0 = 0.$$

We will now use $\gamma_t \in (\ker E_t)^\perp$, $d\gamma_t = 0$, to show that $\gamma_0 = 0$, which is a contradiction. Since $\Lambda^{1,2} = \operatorname{Im} E_t \oplus \ker E_t$, we can write $\gamma_t = E_t(\beta_t)$, and so

$$\gamma_t = \partial \bar{\partial} \beta_1 + \bar{\partial}^\dagger \beta_2 + \partial^\dagger \beta_3.$$

Since $d\gamma_t = 0$, we have
$$0 = (\gamma_t, \bar{\partial}^\dagger \beta_2 + \partial^\dagger \beta_3) = \|\bar{\partial}^\dagger \beta_2 + \partial^\dagger \beta_3\|^2.$$
So $\gamma_t = d\alpha_t$. We note
$$(\gamma_t, \gamma_0)_{L^2(X_0)} = (\alpha_t, d_0^\dagger \gamma_0)_{L^2(X_0)} = 0.$$
Let $t \to 0$, we conclude
$$(\gamma_0, \gamma_0)_{L^2(X_0)} = 0$$
which is a contradiction.

□

Using this lemma, we obtain
$$\|\gamma_t\|_{C^{4,\alpha}} \leqslant C\|\bar{\partial}\chi_t\|_{C^\alpha} = C\|\partial(\Theta_t^* \omega_0)^{0,2}\|_{C^\alpha}.$$
The estimate (3.12) now follows from
$$\|(\Theta_t^* \omega_0)^{0,2}\|_{C^k} \leqslant C_k |t|.$$
This is because $\Theta_0 = id$, $\omega_0 \in \Lambda^{1,1}$, and
$$|(\Theta_t^* \omega_0)^{0,2}| = \left| \int_0^t \frac{d}{ds}(\Theta_s^* \omega_0)^{0,2} ds \right| \leqslant \int_0^t |L_{V_s} \omega_0| ds \leqslant C \int_0^t ds.$$
and similarly for any C^k norm of $(\Theta_t^* \omega_0)^{0,2}$.

We now prove that the forms $\omega_{\gamma_t} \to \omega$ as $t \to 0$.
$$\|\omega_{\gamma_t} - \omega\|_{C^0(X,g_0)} \leqslant \|\chi_t - \omega\|_{C^0(X,g_0)} + C\|\gamma_t\|_{C^3(X,g_0)} \leqslant C|t|.$$
Similarly, $\omega_{\gamma_t} \to \omega$ as $t \to 0$ in any C^k norm.

Corollary 3.10. *Let $\mathcal{X} \to \Delta$ be a family of complex manifolds with X_0 a compact Kähler manifold. Then $h^{p,q}(X_t) = h^{p,q}(X_0)$ for all small enough t.*

Proof. By the Kodaira-Spencer theorem, X_t is a Kähler manifold for all t small enough. By the semi-continuity theorem, $h^{p,q}(X_t) \leqslant h^{p,q}(X_0)$ for small t, so we suppose by contradiction that there exists t and p, q such that $h^{p,q}(X_t) < h^{p,q}(X_0)$. Let $p + q = k$, and
$$\sum_{i+j=k} h^{i,j}(X_t) < \sum_{i+j=k} h^{i,j}(X_0) = b^k = \sum_{i+j=k} h^{i,j}(X_t)$$
which is a contradiction. Here we used that X_0 and X_t are diffeomorphic so they have the same Betti numbers b^k, and the Hodge decomposition $b^k = \sum_{i+j=k} h^{i,j}$ for Kähler manifolds. □

4 Calabi-Yau Threefolds

4.1 Parameters of threefolds

There are various inequivalent definitions of Calabi-Yau manifolds used in the literature. The Wikipedia page for Calabi-Yau manifolds gives some of the commonly used definitions. In these notes, we will use the following definition:

Definition 4.1. *Our definition of a Kähler Calabi-Yau manifold is a simply-connected compact complex manifold of dimension n admitting a Kähler metric ω and a nowhere vanishing holomorphic $(n,0)$ form Ω.*

The section Ω is a nowhere vanishing holomorphic section of the canonical bundle $K_X = (\det T^{1,0}X)^*$, and so $K_X = \mathcal{O}_X$ is the trivial holomorphic bundle.

Let $(Y,\Omega,\hat{\omega})$ be a Kähler Calabi-Yau threefold. Since we assume that Y is simply connected, we have $b_1(X) = 0$ and so by the Hodge decomposition (2.11) then

$$h^{1,0} = h^{0,1} = 0.$$

In addition to the Hodge symmetries

$$h^{p,q} = h^{q,p}, \quad h^{p,q} = h^{3-p,3-q},$$

Calabi-Yau threefolds satisfy

$$h^{0,p} = h^{3,p}.$$

This is just $H^p(X,K_X) = H^p(X,\mathcal{O}_X)$. Therefore, the only Hodge numbers to consider are

$$h^{1,1}, \quad h^{2,1}.$$

By the Hodge decomposition (2.11), we have $b_2 = h^{1,1}$ and $b_3 = 2 + 2h^{2,1}$. The Euler characteristic is defined by

$$\chi(Y) = \sum_{i=1}^{6} (-1)^i b_i,$$

which becomes in this case

$$\chi(Y) = 2(h^{1,1} - h^{2,1}).$$

In summary, for each Calabi-Yau threefold, we associate two parameters $(h^{1,1}, h^{2,1})$.

• $h^{1,1}$ encodes the Kähler classes of X. A Kähler metric ω produces a non-zero class

$$[\omega] \in H^2_{dR}(X,\mathbb{R}) \cap H^{1,1}(X,\mathbb{R}).$$

To see $[\omega] \neq 0 \in H^2_{dR}$, consider $\int_X \omega^n$. If $\omega^n = d\alpha$, then $\int_X \omega^n = 0$. On the other hand, $\omega^n = (\det g)\, idz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge idz^n \wedge d\bar{z}^n$. Since $(\det g) > 0$ at all points, then $\int_X \omega^n > 0$.

Let $\mathcal{C}$ be the set of all $[\alpha] \in H^{1,1}(X,\mathbb{R})$ such that there exists a Kähler metric ω with $\omega \in [\alpha]$. We call $\mathcal{C}$ the Kähler cone. It turns out that $\mathcal{C}$ is an open convex cone in $H^{1,1}(X,\mathbb{R})$ (see e.g. Tosatti's note [28]). Therefore the Kähler cone has real dimension $h^{1,1}$. By the $\partial\bar{\partial}$-Lemma, if $[\omega] \in \mathcal{C}$, then

$$[\omega] = \{\omega + i\partial\bar{\partial}\varphi > 0 \text{ with } \varphi \in C^\infty(X,\mathbb{R})\}$$

and a given Kähler class $[\omega]$ is parametrized by functions.

• $h^{2,1}$ encodes the infinitesimal complex structure deformations of X. We discussed earlier how a 1-parameter family of complex structures $(X, J(t))$ produces an element $[\eta] \in H^1(X, T^{1,0}X)$ by $\eta = \dot{J}(0)$. Note that by Serre duality

$$\dim H^1(T^{1,0}X) = \dim H^2(K_X \otimes (T^{1,0}X)^*) = \dim H^2(\Omega^1) = h^{1,2}.$$

On a Calabi-Yau manifold, the inverse problem can be solved.

Theorem 4.2. *(Bogomolov-Tian-Todorov Theorem) Let (X, J) be a Kähler Calabi-Yau threefold. Let $[\eta] \in H^1(X, T^{1,0}X)$. Then η can be attained by a path of complex structures $(X, J(t))$ such that $J(0) = J$ and $[\dot{J}(0)] = [\eta]$.*

Textbook references for the proof of this include e.g. [14, 16]. Since $\dim H^1(X, T^{1,0}X) = \dim H^2(X, \Omega^1)$ on a Calabi-Yau threefold, the number $h^{2,1}$ is understood as parametrizing deformations of complex structure.

4.2 Ricci flat metrics

Let X be a complex manifold with holomorphic volume form Ω and hermitian metric ω. In local holomorphic coordinates, then

$$\omega = i g_{j\bar{k}}\, dz^j \wedge d\bar{z}^k, \quad \Omega = f dz^1 \wedge \dots dz^n$$

where $f(z)$ is a local nowhere vanishing holomorphic function. The norm of Ω induced on $(\det T^{1,0}X)^*$ is

$$|\Omega|^2_\omega = \frac{f(z)\overline{f(z)}}{\det g_{j\bar{k}}}.$$

The Chern-Ricci curvature $R_{j\bar{k}} = -\partial_{\bar{k}}\partial_j \log\det g$ can also be written

$$R_{j\bar{k}} = \partial_{\bar{k}}\partial_j \log |\Omega|^2_\omega.$$

Indeed,

$$\partial_{\bar{k}}\partial_j \log\det |\Omega|^2_\omega = \partial_{\bar{k}}\partial_j \log |f|^2 - \partial_{\bar{k}}\partial_j \log\det g,$$

and $\partial_{\bar{k}}\partial_j \log |f|^2 = 0$ for any such function f. This is because

$$\partial_j\partial_{\bar{k}} \log |f|^2 = \partial_j \frac{f \partial_{\bar{k}} \bar{f}}{f\bar{f}} = \overline{\partial_{\bar{j}} \frac{\partial_k f}{f}} = 0.$$

The hermitian metric ω is Chern-Ricci flat if $R_{j\bar{k}} = 0$ which implies

$$0 = g^{j\bar{k}} \partial_j \partial_{\bar{k}} \log |\Omega|^2_\omega.$$

The maximum principle on compact manifolds implies that $|\Omega|_\omega$ is constant. Here is a quick proof if ω is Kähler. (Note: this PDE result holds without the Kähler assumption. The general proof uses the Hopf maximum principle instead of integration by parts. The general statement is: if $a^{ij}\partial_i\partial_j f + b^i \partial_i f = 0$ on a compact manifold with a^{ij} positive-definite, then f is constant.)

First, note the identities for a real function $u : X \to \mathbb{R}$:

$$i\partial\bar{\partial}u \wedge \omega^{n-1} = \frac{1}{n}(g^{j\bar{k}}u_{j\bar{k}})\omega^n, \quad u_{j\bar{k}} = \partial_j\partial_{\bar{k}}u,$$

and

$$i\partial u \wedge \bar{\partial}u \wedge \omega^{n-1} = \frac{1}{n}|\partial u|^2_g\omega^n.$$

These can be checked at a point $p \in X$ with $\omega|_p = \sum_k idz^k \wedge d\bar{z}^k$. By Stokes's theorem and $d\omega = 0$,

$$\int_X (\log|\Omega|^2_\omega)id\bar{\partial}\log|\Omega|^2_\omega \wedge \omega^{n-1} = -\int_X d(\log|\Omega|^2_\omega) \wedge i\bar{\partial}\log|\Omega|^2_\omega \wedge \omega^{n-1}.$$

which implies

$$0 = \int_X (\log|\Omega|^2_\omega)(g^{j\bar{k}}\partial_j\partial_{\bar{k}}\log|\Omega|^2_\omega)\,\omega^n = -\int_X |\partial\log|\Omega|^2_\omega|^2_g\,\omega^n$$

and so $|\partial\log|\Omega|^2_\omega|^2_g = 0$ and $\log|\Omega|^2_\omega$ is a constant.

In summary, on a compact complex manifold with trivial canonical bundle, then $R_{j\bar{k}} = 0$ is equivalent to $|\Omega|^2_\omega = const$. To find such metrics, we fix ω an arbitrary hermitian metric, and look for solutions to this equation of the form

$$\tilde{g}_{j\bar{k}} = g_{j\bar{k}} + \partial_j\partial_{\bar{k}}\varphi > 0$$

or in differential form notation

$$\tilde{\omega} = \omega + i\partial\bar{\partial}\varphi > 0.$$

The equation $|\Omega|^2_{\tilde{\omega}}(x) = e^b$, $b \in \mathbb{R}$, can be written

$$e^b = \frac{|\Omega|^2_{\tilde{\omega}}}{|\Omega|^2_\omega}e^{\log|\Omega|^2_\omega} = \left[\frac{\det g}{\det\tilde{g}}\right]e^{\log|\Omega|^2_\omega},$$

which leads to the complex Monge-Ampère equation

$$\left[\frac{\det(g_{j\bar{k}} + \varphi_{j\bar{k}})}{\det g_{j\bar{k}}}\right] = e^{\log|\Omega|^2_\omega - b}.$$

When $d\omega = 0$, the constant b can be identified from the initial data (ω, Ω), since the equation can also be written

$$(\omega + i\partial\bar{\partial}\varphi)^n = e^{-b}|\Omega|^2_\omega\omega^n$$

and integration of both sides and Stokes's theorem gives

$$\int_X \omega^n = \int_X e^{-b}|\Omega|^2_\omega\omega^n.$$

Indeed:

$$\begin{aligned}
\int_X (\omega + i\partial\bar{\partial}\varphi)^n &= \int_X \omega^n + \sum_{k=1}^n c_k\int_X \omega^{n-k} \wedge (i\partial\bar{\partial}\varphi)^k \\
&= \int_X \omega^n + \sum_{k=1}^n c_k\int_X d(\omega^{n-k} \wedge (i\bar{\partial}\varphi) \wedge (i\partial\bar{\partial}\varphi)^{k-1}) \\
&= \int_X \omega^n. \qquad (4.1)
\end{aligned}$$

Another way to write the Kähler-Ricci flat metric equation is

$$(\omega + i\partial\bar{\partial}\varphi)^n = e^{-b} i^{n^2} \Omega \wedge \bar{\Omega},$$

because of the identity

$$i^{n^2} \Omega \wedge \bar{\Omega} = |\Omega|^2_\omega \frac{\omega^n}{n!}.$$

Yau's theorem states:

Theorem 4.3. *[31] Let (X, ω) be a compact Kähler manifold. Let e^h be an arbitrary function and b be the constant $e^b = \int_X e^h \omega^n / \int_X \omega^n$. Then there exists a unique smooth solution $u : X \to \mathbb{R}$ solving*

$$(\omega + i\partial\bar{\partial}u)^n = e^{h-b}\omega^n, \quad \omega + i\partial\bar{\partial}u > 0$$

and $\int_X u\, \omega^n = 0$.

References for the proof of this theorem include: Chapter 2 of Siu's notes [24] and Chapter 3 of G. Szekelyhidi's book [26].

The complex Monge-Ampère equation can also be solved for a pair (u, b) on a general hermitian manifold (X, ω): this theorem is due to Tosatti-Weinkove [29].

As a consequence of Yau's theorem, a Kähler Calabi-Yau manifold admits Kähler Ricci-flat metrics. These are also called Calabi-Yau metrics ω_{CY} and they solve

$$\mathrm{Ric}(\omega_{CY}) = 0, \quad d\omega_{CY} = 0.$$

Furthermore, each Kähler class $[\omega] \in H^{1,1}(X, \mathbb{R})$ contains a unique Calabi-Yau representative $\omega_{CY} \in [\omega]$ by solving the Monge-Ampère equation with ansatz $\omega + i\partial\bar{\partial}\varphi$.

4.3 Deformations of complex structure

By the BTT theorem, on a Calabi-Yau threefold X then any element $[\eta] \in H^1(X, T^{1,0}X)$ can be attained by a family of complex manifolds. Here we note that a deformation of a Calabi-Yau threefold also carries a Calabi-Yau structure.

Proposition 4.4. *Let (X, J, ω, Ω) be a Kähler Calabi-Yau threefold. Let $(X, J(t))$ be a smooth path of complex structures with $J(0) = J$ and $|t| < \varepsilon$. For $\varepsilon > 0$ small enough, then there exists a family $(\omega(t), \Omega(t))$ with $(\omega(t), \Omega(t)) \to (\omega, \Omega)$ as $t \to 0$ such that $\omega(t)$ is a Kähler metric and $\Omega(t)$ is a nowhere vanishing holomorphic volume form on $(X, J(t))$.*

Proof. The existence of the family of Kähler metrics $\omega(t) = \omega_t$ with $\omega_t \to \omega$ is Kodaira-Spencer's stability theorem [21] which we discussed earlier. We now describe how to construct $\Omega(t) = \Omega_t$. The 3-form Ω is defined on $(X, J(t))$, however it does not necessarily have type $(3,0)$. So we take the $(3,0)$ part $(\Omega)^{3,0}_t$, and this is nowhere vanishing for small t by continuity of the complex structures $J(t)$. We now need to correct $(\Omega)^{3,0}_t$ to make it holomorphic. Write

$$(\Omega)^{3,0}_t = \sigma_t dz^1_t \wedge dz^2_t \wedge dz^3_t.$$

Note $\alpha = \sigma_t^{-1}\bar{\partial}_t\sigma_t$ is a well-defined $(0,1)$-form on X_t. This is because of the transformation law $\sigma \mapsto t_{UV}\sigma$ with t_{UV} holomorphic, which implies $\alpha_{\bar{k}} \mapsto \alpha_{\bar{k}}$. We also note that $\bar{\partial}\alpha = 0$, since

$$\bar{\partial}\alpha = -\sigma^{-2}\bar{\partial}\sigma \wedge \bar{\partial}\sigma + \sigma^{-1}\bar{\partial}^2\sigma = 0.$$

We claim that we can find a smooth function u_t such that

$$\bar{\partial}_t u_t = -\alpha_t, \quad \int_{X_t} u\,\omega_t^n = 0$$

This is because X is assumed to be simply-connected, and so $H^1(X, \mathcal{O}_X) = 0$. By the semi-continuity theorem, $H^1(X_t, \mathcal{O}_{X_t}) \leqslant H^1(X, \mathcal{O}_X)$, hence $H^1(X_t, \mathcal{O}_{X_t}) = 0$. The definition of Dolbeault cohomology states that the closed $(0,1)$-form α comes from $\bar{\partial}$ of a function. Using u, we define

$$\Omega_t = e^{u_t}\sigma_t dz_t^1 \wedge dz_t^2 \wedge dz_t^3.$$

This solves $\bar{\partial}\Omega_t = 0$ since

$$\partial_{\bar{k}}(e^u\sigma) = e^u\partial_{\bar{k}}u\sigma + e^u\partial_{\bar{k}}\sigma = e^u(-\sigma^{-1}(\partial_{\bar{k}}\sigma)\sigma + \partial_{\bar{k}}\sigma) = 0.$$

Therefore Ω_t defines a nowhere vanishing holomorphic volume form on (X, J_t). Next, we show $\Omega_t \to \Omega$. For this, we show

$$|u_t| \leqslant C|t|. \tag{4.2}$$

Then since $\sigma_t \to \sigma$ smoothly,

$$\begin{aligned} |e^u\sigma_t - \sigma| &\leqslant |\sigma_t||e^u - 1| + |\sigma_t - \sigma| \\ &\leqslant C|e^{C|t|} - 1| + C|t| \\ &\leqslant C\sum_{k=1}^{\infty}\frac{1}{k!}(C|t|)^k + C|t| \leqslant C|t|. \end{aligned} \tag{4.3}$$

Hence $|\Omega_t - \Omega|_g \leqslant C|t|$. To prove (4.2), we will use the complex Laplacian

$$\Delta_{g_t} : C^\infty(X, \mathbb{R}) \to C^\infty(X, \mathbb{R}),$$

given by

$$\Delta_{g_t} f = (g_t)^{j\bar{k}}\partial_j\partial_{\bar{k}}f.$$

We proved earlier that $\ker\Delta_{g_t} = \mathbb{R}\cdot 1$ are constants. Therefore since $\int_{X_t} u_t\,\omega_t^n = 0$, we have that $u_t \in \ker(\Delta_{g_t})^\perp$ with respect to the g_t inner product for all t. By elliptic estimates,

$$\|u_t\|_{C^{2,\alpha}(X,g)} \leqslant C\|\Delta_{g_t}u_t\|_{C^\alpha(X,g)}.$$

One should verify that the constant C is uniform in t. We omit the proof, but it follows an outline similar to (3.13): the usual Schauder estimates $\|u\|_{C^{2,\alpha}} \leqslant C(\|u\|_{L^\infty} + \|\Delta u\|_{C^\alpha})$ are uniform in t, and to remove the extra $\|u\|_{L^\infty}$ we assume the estimate fails as $t \to 0$ and derive a contradiction.

Therefore since $\partial_{\bar{k}}u = \sigma_t^{-1}\partial_{\bar{k}}\sigma_t$

$$\|u_t\|_{C^{2,\alpha}(X,g)} \leqslant C\|g^{j\bar{k}}\partial_j(\sigma_t^{-1}\partial_{\bar{k}}\sigma_t)\| \leqslant C|t|$$

since σ_t tends smoothly to the holomorphic σ with $\bar{\partial}\sigma = 0$.

□

Let $(X, J(t), \omega(t), \Omega(t))$ be a family of Kähler Calabi-Yau threefolds with $\dot{J}(0) = \eta$. We use the notation $\delta = \frac{d}{dt}|_{t=0}$. To end this section, we will compute the variation

$$\chi := \delta\Omega$$

and relate it to η via

$$\chi_{\bar{\alpha}\beta\gamma} = \frac{i}{2}\bigg[- \eta^{\mu}{}_{\bar{\alpha}}\Omega_{\mu\beta\gamma}\bigg].$$

Here everything is evaluated at holomorphic coordinates with respect to the initial (X, J, Ω). We will show $\chi \in \Lambda^{2,1}(X, J)$, and since $\bar{\partial}\eta = 0$, it follows that $\bar{\partial}\chi = 0$.

We start by differentiating

$$J^r{}_i J^s{}_j J^t{}_k \Omega_{rst} = -i\Omega_{ijk} \tag{4.4}$$

to obtain

$$- i\delta\Omega_{ijk} = \delta J^r{}_i J^s{}_j J^t{}_k \Omega_{rst} + J^r{}_i \delta J^s{}_j J^t{}_k \Omega_{rst} + J^r{}_i J^s{}_j \delta J^t{}_k \Omega_{rst} + J^r{}_i J^s{}_j J^t{}_k \delta\Omega_{rst} \tag{4.5}$$

In holomorphic coordinates α, β, γ, then $J^{\alpha}{}_{\beta} = i\delta^{\alpha}{}_{\beta}$, $J^{\bar{\alpha}}{}_{\bar{\beta}} = -i\delta^{\alpha}{}_{\beta}$, and the only non-zero components of Ω are unbarred, which implies

$$\delta\Omega_{\bar{\alpha}\bar{\beta}\bar{\gamma}} = 0, \quad \delta\Omega_{\bar{\alpha}\bar{\beta}\gamma} = 0 \tag{4.6}$$

and

$$- i\delta\Omega_{\bar{\alpha}\beta\gamma} = -\delta J^r{}_{\bar{\alpha}}\Omega_{r\beta\gamma} + \delta J^s{}_j \Omega_{\bar{\alpha}s\gamma} + \delta J^t{}_k \Omega_{\bar{\alpha}\beta t} + (-i)i^2\delta\Omega_{\bar{\alpha}\beta\gamma} \tag{4.7}$$

which implies

$$\chi_{\bar{\alpha}\beta\gamma} := \delta\Omega_{\bar{\alpha}\beta\gamma} = \frac{i}{2}\bigg[- \delta J^{\mu}{}_{\bar{\alpha}}\Omega_{\mu\beta\gamma}\bigg]. \tag{4.8}$$

We can also invert the formula for χ to get δJ from $\delta\Omega$. At a point where $g_{ij} = \delta_{ij}$, then

$$\Omega = f(z)dz^1 \wedge dz^2 \wedge dz^3, \quad |\Omega|^2_{\omega} = |f(z)|^2, \tag{4.9}$$

and

$$\bar{\Omega}^{\beta\gamma\nu}\bigg[- \frac{i}{2}\Omega_{\mu\beta\gamma}\delta J^{\mu}{}_{\bar{\alpha}}\bigg] = -i|\Omega|^2_{\omega}\delta J^{\nu}{}_{\bar{\alpha}}. \tag{4.10}$$

Thus

$$\delta J^{\nu}{}_{\bar{\alpha}} = \frac{i}{|\Omega|^2_{\omega}}\bar{\Omega}^{\beta\gamma\nu}\delta\Omega_{\bar{\alpha}\beta\gamma} \tag{4.11}$$

or

$$\eta^{\nu}{}_{\bar{\alpha}} = \frac{i}{|\Omega|^2_{\omega}}\bar{\Omega}^{\beta\mu\nu}\chi_{\bar{\alpha}\beta\mu}. \tag{4.12}$$

4.4 Quintic threefolds

4.4.1 Holomorphic volume form

Consider $X \subset \mathbb{P}^4$ given by $\{P = 0\}$, where

$$P = x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5.$$

In the coordinate chart $U_0 = \{x_0 \neq 0\}$ with coordinates $z_i = z_i/x_0$, X appears as

$$\{f = 1 + z_1^5 + z_2^5 + z_3^5 + z_4^5 = 0\} \subset \mathbb{C}^4.$$

Since

$$Df = [5z_1^4, 5z_2^4, 5z_3^4, 5z_4^4]$$

we see that Df has maximal rank except at $z = 0$, which is not included in the set $\{f = 0\}$. Therefore X is a smooth complex manifold of complex dimension three. This manifold admits a holomorphic volume form Ω. This can be seen by the adjunction formula, but also by the explicit expression

$$\Omega|_{U_0 \cap V_4} = \frac{dz_1 \wedge dz_2 \wedge dz_3}{\partial f/\partial z_4} \tag{4.13}$$

over the set $U_0 = \{x_0 \neq 0\}$ intersected with $V_4 = \{\partial f/\partial z_4 \neq 0\}$. We now verify that Ω extends from $U_0 \cap V$ to a holomorphic volume form on all of X.

• Extending from V_4 to V_3. On the intersection $V_3 \cap V_4$, by the implicit function theorem we can write $z_3 = g(z_1, z_2, z_4)$. Therefore

$$\Omega = \frac{dz_1 \wedge dz_2 \wedge dg}{\partial f/\partial z_4} = \frac{\partial g/\partial z_4}{\partial f/\partial z_4} dz_1 \wedge dz_2 \wedge dz_4.$$

Differentiating $f(z_1, z_2, g(z_1, z_2, z_4), z_4) = 0$ in z_4 gives

$$\partial_3 f \partial_4 g + \partial_4 f = 0.$$

Hence

$$\Omega|_{U_0 \cap V_3} = -\frac{dz_1 \wedge dz_2 \wedge dz_4}{\partial f/\partial z_3}$$

This is holomorphic and nowhere vanishing on V_3, therefore Ω extends from V_4 to V_3. Similar arguments show that Ω defines a holomorphic volume form on all of U_0. Next, we need to extend Ω beyond $U_0 \subset X$.

• Extending from U_0 to U_1. On U_0 we have coordinates $z_i = x_i/x_0$, and on U_1 we have coordinates $w_1 = x_0/x_1$, and for $i \geqslant 2$ then $w_i = x_i/x_1$. The change of coordinates on $U_0 \cap U_1$ is then

$$z_1 = w_1^{-1}, \quad z_2 = w_1^{-1} w_2, \quad z_3 = w_1^{-1} w_3, \quad z_4 = w_1^{-1} w_4.$$

The holomorphic volume form on $U_0 \cap U_1 \cap V_4$ becomes

$$\Omega = \frac{dw_1^{-1} \wedge d(w_1^{-1} w_2) \wedge d(w_1^{-1} w_3)}{5z_4^4}$$

since $\partial f/\partial z_4 = 5z_4^4$, and therefore

$$\Omega = -\frac{dw_1 \wedge dw_2 \wedge dw_3}{5w_4^4}.$$

Over U_1, $\{P = 0\}$ appears as $\{\tilde{f} = 1 + w_1^5 + w_2^5 + w_3^5 + w_4^5 = 0\}$, and so

$$\Omega|_{U_1 \cap \tilde{V}_4} = -\frac{dw_1 \wedge dw_2 \wedge dw_3}{\partial \tilde{f}/\partial w_4}.$$

As before, Ω extends from $U_1 \cap \tilde{V}_4$ to all of U_1, and similarly Ω extends to a nowhere vanishing holomorphic form on U_2, U_3, U_4.

Putting everything together, we see that the local expression (4.13) defines a holomorphic volume form on all of X.

4.4.2 Hodge numbers

The Hodge numbers of the quintic threefold X are

$$h^{1,1} = 1, \quad h^{1,2} = 101.$$

• $h^{1,1} = 1$. This follows from the Lefschetz hyperplane theorem, which we recall states: let $Y \subseteq X$ be a complex hypersurface such that its associated line bundle $\mathcal{O}(Y) \to X$ is positive. Then

$$h^{p,q}(Y) = h^{p,q}(X), \quad p + q \leqslant n - 2.$$

We computed that for $Y = \{P = 0\} \subset \mathbb{P}^n$ with P of degreen k, then $\mathcal{O}(Y) = \mathcal{O}(k)$. We also computed that the Fubini-Study metric on $\mathcal{O}(1)$ has positive curvature, hence $\mathcal{O}(k) = [\mathcal{O}(1)]^k$ is also a positive bundle. Therefore the Lefschetz hyperplane theorem applies to the quintic $Y \subset \mathbb{P}^4$, and

$$h^{1,1}(Y) = h^{1,1}(\mathbb{P}^4).$$

We computed earlier that $h^{1,1}(\mathbb{P}^n) = 1$, and so $h^{1,1} = 1$ for the quintic.

• $h^{1,2} = 101$. We only give here a heuristic argument from the string theory literature, but a real proof can be found in e.g. [23]. We discussed earlier that $h^{1,2}$ parametrizes complex structure deformations. We can deform the complex structure of the quintic

$$\sum_{i=0}^{4} Z_i^5 = 0$$

by introducing parameters $c_I = c_{i_0 i_1 i_2 i_3 i_4}$

$$\sum_{|I|=5} c_I Z_I = 0, \quad Z_I = Z_0^{i_0} Z_1^{i_1} Z_2^{i_2} Z_3^{i_3} Z_4^{i_4}. \tag{4.14}$$

The number of parameters are: number of ways of placing 5 objects (the powers) in 5 bins (the Z_i), which is $(5+4)!/5!4! = 126$. But some of these 126 coefficients do not give genuine deformations of complex structure. There are 5^2 degrees of freedom coming from matrices $A \in GL(5)$ which produces a biholomorphism

$$A : \{P(Z) = 0\} \subseteq \mathbb{P}^4 \to \{P(AZ) = 0\} \subseteq \mathbb{P}^4.$$

So we are left with $126 - 25 = 101$ parameters, which matches up with $h^{1,2} = 101$. This is not a proof because a priori there could be ways to deform the complex structure of a quintic which is not by (4.14).

4.4.3 Nodal singularities

We now the quintic by a parameter $t \in \mathbb{C}$ and consider

$$X_t = \left\{ Q_t = \sum_{i=0}^{4} Z_i^5 - 5tZ_0Z_1Z_2Z_3Z_4 = 0 \right\} \subseteq \mathbb{P}^4.$$

We note in passing that this family was used by [3] to construct one of the first examples of mirror symmetry. We first notice that at $t = 1$, this is no longer a smooth manifold as it contains singular points. To find the singular points, we set all derivatives of Q_t to zero.

$$5Z_k^4 = 5t \prod_{i \neq k} Z_i$$

and so

$$\left(\prod_{i=0}^{4} Z_i \right)^4 = t^5 \left(\prod_{i=0}^{4} Z_i \right)^4$$

If one of the $Z_i = 0$, then they all are, which is not a point in projective space. So we conclude that singular points occur when $t^5 = 1$.

We let $t = 1$ and investigate the singular quintic which we denote $\underline{X}$. There are 125 singular points: these occur when $Z_0^5 = Z_1^5 = \cdots = Z_4^5 = \prod Z_i$. Dividing the singular points by $\prod Z_i$ (in projective space), singular points are given by roots of unity $Z_k^5 = 1$.

$$q = [\zeta^{\alpha_0}, \ldots, \zeta^{\alpha_4}], \quad \zeta^5 = 1$$

with $\alpha_i \in \{0, 1, 2, 3, 4\}$. Since $q \in X$, we must have $\sum_i \alpha_i = 0 \bmod 5$. We can always represent q in projective space with the first entry equal to 1, so that leaves 3 free parameters α_1, α_2, α_3 with α_4 determined by $\sum_i \alpha_i = 0$, so there are $(5)(5)(5) = 125$ singular points.

We now look locally near the point $q = [1, 1, 1, 1, 1]$. In the coordinate chart (U_0, z), $q = (1, 1, 1, 1)$ and the equation of the singular quintic is

$$f(z) = 1 + \sum_{i=1}^{4} z_i^5 - 5z_1z_2z_3z_4 = 0.$$

The holomorphic function f satisfies $f(q) = 0$ and $Df(q) = 0$. Its holomorphic Hessian matrix

$\frac{\partial^2 f}{\partial z^i \partial z^j}$ at q is

$$\begin{bmatrix} 20 & -5 & -5 & -5 \\ -5 & 20 & -5 & -5 \\ -5 & -5 & 20 & -5 \\ -5 & -5 & -5 & 20 \end{bmatrix}$$

This is a non-singular matrix. Thus we have a holomorphic function $f(z_1, \ldots, z_4)$ with $f(q) = 0$, $df(q) = 0$, but $D^2 f(q)$ is non-degenerate. There is a holomorphic Morse lemma (e.g. Lemma 2.11 in the book [30]) which gives the existence of holomorphic coordinates w with $q(w) = 0$ such that

$$f(w) = \sum_{i=1}^{4} w_i^2.$$

We give the proof in complex dimension $n = 2$ following (Lemma 42 p.242 in [8]). First, shift coordinates so that $q = 0$ and $f(z_1, z_2)$ is a holomorphic function with $f(0) = 0$, $Df(0) = 0$, and $\partial_i \partial_j f(0)$ a non-degenerate symmetric matrix. For a symmetric complex matrix A, there is a unitary matrix U such that $UAU^T = D$ where D is diagonal (Autonne–Takagi factorization). Since A is non-degenerate the diagonal elements are non-zero, so we may multiply on both sides by a diagonal complex matrix to form $SAS^T = I$.

So we can write $S^T[D^2 f(0)]S = I$ for a complex matrix S, and let $z^k = S^i{}_k y^k$. By the chain rule

$$\frac{\partial f}{\partial y^k} = \frac{\partial f}{\partial z^p} \frac{\partial z^p}{\partial y^k} = S^p{}_k f_p, \quad \frac{\partial^2 f}{\partial y^j \partial y^k} = S^p{}_k f_{pq} S^q{}_j.$$

Written in matrix notation, this is

$$D_y^2 f(0) = S^T[D^2 f(0)]S = I.$$

Hence we may assume that the power series of f is

$$f = z_1^2 + z_2^2 + \sum_{i+j>2} a_{ij} z_1^i z_2^j.$$

Let $f_2 = \frac{\partial f}{\partial z_2}$. Since $\partial_2 f_2(0) \neq 0$, by the holomorphic implicit function theorem there is a function $a(z_1)$ with $a(0) = 0$ such that $f_2(z_1, a(z_1)) = 0$. Differentiating this gives $a'(0) = 0$. Define new coordinates by

$$\tilde{z}_1 = z_1, \quad \tilde{z}_2 = z_2 - a(z_1)$$

and let $\tilde{f}(\tilde{z}_1, \tilde{z}_2) = f(z_1, z_2 - a(z_1))$. We will compute the power series of $\tilde{f}$. We start with 1st derivatives:

$$\partial_1 \tilde{f} = \partial_1 f - a' \partial_2 f, \quad \partial_2 \tilde{f} = \partial_2 f.$$

The key observation is that $\partial_2 \tilde{f} \equiv 0$ when $\tilde{z}_2 = 0$, and so

$$\partial_2 \tilde{f} = \tilde{z}_2 Q(\tilde{z}_1, \tilde{z}_2). \tag{4.15}$$

Next, we move on to second order derivatives.

$$\partial_1 \partial_2 \tilde{f} = \tilde{z}_2 \partial_1 Q(\tilde{z}_1, \tilde{z}_2),$$

$$\partial_1\partial_1\tilde{f} = \partial_1\partial_1 f - 2a'\partial_1\partial_2 f + (a')^2\partial_2\partial_2 f - a''\partial_2 f$$
$$\partial_2\partial_2\tilde{f} = \partial_2\partial_2 f.$$

Evaluating all these at zero, and using that (4.15) implies $\partial_1^k\partial_f f(0) = 0$, we get the expansion

$$\begin{aligned} f &= \tilde{z}_1^2 + \sum_{k\geqslant 3} b_k\tilde{z}_1^k + \tilde{z}_2^2 + \sum_{i+j\geqslant 3,\ j\geqslant 2} a_{ij}\tilde{z}_1^i\tilde{z}_2^j \\ &= \tilde{z}_1^2\left(1+\sum_{k\geqslant 1} b_k\tilde{z}_1^k\right) + \tilde{z}_2^2\left(1+\sum_{i+j\geqslant 3,\ j\geqslant 0} a_{ij}\tilde{z}_1^i\tilde{z}_2^j\right) \end{aligned} \tag{4.16}$$

We can now let

$$w_1 = \tilde{z}_1\left(1+\sum_{k\geqslant 1} b_k\tilde{z}_1^k\right)^{1/2}, \quad w_2 = \tilde{z}_2\left(1+\sum_{i+j\geqslant 3,\ j\geqslant 0} a_{ij}\tilde{z}_1^i\tilde{z}_2^j\right)^{1/2},$$

using that a holomorphic function $\psi : \mathbb{C} \to \mathbb{C}$ with $\psi(0) \neq 0$ has a local square root defined in a neighborhood of the origin. In these new coordinates then, $f = w_1^2 + w_2^2$.

In summary, the singular quintic $\underline{X}$ has holomorphic coordinates around each singular point where the singularity appears as

$$\left\{\sum_{i=1}^4 z_i^2 = 0\right\} \subseteq \mathbb{C}^4.$$

Such singularities are called nodes, or nodal singularities, or ordinary double points (ODP).

4.4.4 Examples of conifold transitions

Example 4.5. There are 2 ways to desingularize

$$\underline{X} = \left\{\sum_{i=0}^4 Z_i^5 - 5\prod_{i=0}^4 Z_i = 0\right\} \subseteq \mathbb{P}^4$$

which is the singular quintic discussed in the previous section.

- The first way is called a smoothing, which is to realize $\underline{X}$ as the central fiber of

$$X_t = \left\{\sum_{i=0}^4 Z_i^5 - (5+t)Z_0Z_1Z_2Z_3Z_4 = 0\right\} \subset \mathbb{P}^4. \tag{4.17}$$

We discussed earlier how for $t \neq 0$, the space X_t no longer has singular points. In the chart (U_0, z), we see the zero set

$$f_t(z) = 1 + \sum_{i=1}^4 z_i^5 - 5z_1z_2z_3z_4 - tz_1z_2z_3z_4 = 0.$$

Near $(1,1,1,1)$, we apply the holomorphic Morse lemma to

$$g(z) = \frac{1+\sum_{i=1}^4 z_i^5}{z_1z_2z_3z_4} - 5,$$

and write $g(w) = \sum_i w_i^2$ (here $w = 0$ corresponds to the point $(1,1,1,1)$) so that the zero set $\{f_t = 0\}$ becomes

$$\{\sum_{i=1}^{4} w_i^2 = t\} \subset \mathbb{C}^4. \tag{4.18}$$

for new coordinates w_i. Locally, the ODP singularities $\sum_i w_i^2 = 0$ has been replaced by $\sum_i w_i^2 = t$.

• The second way to desingularize $\underline{X}$ is by small resolution. We will discuss how this is done in full detail in the next section, but it results in a map

$$\sigma : X \to \underline{X}$$

where X is a complex manifold, $\sigma^{-1}(p) = \mathbb{P}^1$ for each singular point p, and σ is a biholomorphism outside the singular set on X. In other words, each singular point of $\underline{X}$ is replaced by $\mathbb{P}^1$.

The holomorphic volume form Ω on $\underline{X}$ defines a holomorphic volume form $\hat{\Omega} = \sigma^*\Omega$ defined on $X \backslash E$ with $E = \cup \mathbb{P}^1$. By Hartog's theorem, $\hat{\Omega}$ extends to all of X and so X has trivial canonical bundle. We will show later that $\chi(X) - 2N = \chi(X_t)$ where N is the number of nodes. Therefore the two threefolds have different topologies.

Note Hartog's theorem (e.g. p.46 in [7]) states: Let X be a complex manifold. Let $S \subset X$ be a closed complex submanifold of complex codimension $\geqslant 2$: this means there are local coordinates where $U \cap S$ is given by $z_1 = \cdots = z_p = 0$ for $p \geqslant 2$. Then every holomorphic function f on $X \backslash S$ extends holomorphically to X. This is major difference with complex analysis in $\mathbb{C}$: $f(z) = 1/z$ does not extend on $\mathbb{C}\backslash\{0\}$, but any holomorphic $f(z_1, z_2)$ extends on $\mathbb{C}^2\backslash\{0\}$.

Example 4.6. Here is another example from Greene-Morrison-Strominger [12] (see also the exposition in [23]). Define a singular quintic $\underline{X} \subset \mathbb{P}^4$ by the polynomial

$$P = Z_3\, G(Z_0, \ldots, Z_4) + Z_4\, H(Z_0, \ldots, Z_4) = 0. \tag{4.19}$$

where $G = Z_3^4 + Z_2^4 - Z_0^4$ and $H = -Z_4^4 - Z_1^4 - Z_0^4$. The singular points are where $DP = 0$, which happens when

$$Z_3 = 0,\ Z_4 = 0,\ G = 0,\ H = 0. \tag{4.20}$$

There are 16 singular points. (This is also expected by Bezout's theorem: n homogeneous polynomials of degrees d_i in projective space of dimension n has $d_1 \cdot d_n$ intersection points generically. $16 = (1)(1)(4)(4)$ singular points.) We now look at the local model for these singularities. Suppose $p \in \underline{X}$ is a singular point with $p \in U_0 = \{Z_0 \neq 0\}$. In coordinates $z_i = Z_i/Z_0$ the equation for $\underline{X}$ is

$$z_3 g(z) + z_4 h(z) = 0. \tag{4.21}$$

Since $g(z)$ has surjective derivative, by the implicit function theorem there is a holomorphic change of coordinates

$$w_1 = g(z_1, z_2, z_3, z_4),\ w_{i+1} = z_{i+1} \tag{4.22}$$

with inverse $z_1 = \varphi(w_1, w_2, w_3, w_4)$. We can repeat this for $h(z)$ and obtain coordinates $\tilde{w}_i$ with $\tilde{w}_2 = h(w_1, w_2, w_3, w_4)$. Then in these coordinates, the equation for $\underline{X}$ is a neighborhood of the origin in

$$\{\tilde{w}_3\tilde{w}_1 + \tilde{w}_4\tilde{w}_2 = 0\} \subset \mathbb{C}^4. \tag{4.23}$$

This is the model for a nodal singularity. There is another change of coordinates

$$z_1 = \frac{\tilde{w}_1 + i\tilde{w}_2}{2}, \quad z_2 = \frac{\tilde{w}_1 - i\tilde{w}_2}{2}, \quad z_3 = \frac{\tilde{w}_3 + i\tilde{w}_4}{2}, \quad z_4 = \frac{w_3 - iw_4}{2}. \tag{4.24}$$

such that this singularity is represented by

$$\{\sum_{i=1}^{4} z_i^2 = 0\} \subset \mathbb{C}^4. \tag{4.25}$$

One way to resolve $\underline{X}$ is to consider the family of quintics

$$X_t = \{Z_3 G + Z_4 H = t Z_0 Z_1 Z_2 Z_3 Z_4\} \subseteq \mathbb{P}^4.$$

which is now non-singular for $t \neq 0$. As a quintic, it has $h^{1,1}(X_t) = 1$.

Another way is by small resolution. Let $X \subset \mathbb{P}^4 \times \mathbb{P}^1$ given by

$$UZ_4 - VZ_3 = 0, \quad UG(Z) + VH(Z) = 0$$

with $[U : V] \in \mathbb{P}^1$, $[Z_0 : \cdots : Z_4] \in \mathbb{P}^4$. There is a map

$$\sigma : X \to \underline{X}, \quad \sigma([Z_0 : \cdots : Z_4], [U : V]) = [Z_0 : \cdots : Z_4].$$

(check lands in $\underline{X}$). When $P \in \mathrm{Sing}(\underline{X})$ is a singular point $Z_3 = Z_4 = G = H = 0$, then $\sigma^{-1}(P)$ is a full $\mathbb{P}^1$. Otherwise if one of these is non-zero, there is a single point in $\sigma^{-1}(P)$. So we have a map such that $\sigma^{-1}(p_i) = C_i \cong \mathbb{P}^1$ for $p_i \in \mathrm{Sing}(\underline{X})$ and σ^{-1} is a biholomorphism on $\underline{X} \backslash \mathrm{Sing}(\underline{X})$. There are 16 exceptional curves C_i, and we notice that they are not linearly independent in homology.

$$\sum_i a_i [C_i] = 0 \quad \text{in } H_2(X, \mathbb{C}).$$

This is because $b_2(X) = h^{1,1}(X) = 2$. To show this, we use the Lefschetz hyperplane theorem applied to

$$X \subset X' = \{UZ_4 - VZ_3 = 0\} \subset \mathbb{P}^4 \times \mathbb{P}^1.$$

Let $P = UG(Z) + VH(Z)$ so $X = \{P = 0\} \subseteq X'$. Then $\mathcal{O}(X) \to X'$ is $[\mathcal{O}_{\mathbb{P}^4}(4) \otimes \mathcal{O}_{\mathbb{P}^1}(1)]|_{X'} \to X'$. Let's verify this in two specific charts. Take a chart U_1 with coordinates $u = U/V \in \mathbb{P}^1$ and $z = Z/Z_0$ in $\mathbb{P}^4$, and chart U_2 with $v = V/U$ and $w = Z/Z_4$. In chart U_1, the defining equation of X is in coordinates:

$$p_1(u, z) = P/VZ_0^4 = 0,$$

and in chart U_2 the defining equation is $p_2 = P/UZ_4^4$. The cocycle of $\mathcal{O}(X)$ on $U_1 \cap U_2$ is by definition $p_1/p_2 = UZ_4^4/VZ_0^4$, which is the cocycle of $\mathcal{O}_{\mathbb{P}^4}(4) \otimes \mathcal{O}_{\mathbb{P}^1}(1)$.

By the Lefschetz hyperplane theorem, we conclude

$$h^{1,1}(X) = h^{1,1}(X').$$

Similarly, we can show

$$h^{1,1}(X') = h^{1,1}(\mathbb{P}^4 \times \mathbb{P}^1).$$

By the Kunneth formula, $b_2(\mathbb{P}^4 \times \mathbb{P}^1) = 2$. Since $b_2 = h^{1,1} + 2h^{2,0}$ and pullbacks of Kähler metrics give $h^{1,1} \geqslant 2$, we have $h^{1,1}(\mathbb{P}^4 \times \mathbb{P}^1) = 2$. So

$$h^{1,1}(X) = 2.$$

5 Conifold Transitions: Local Model

5.1 Blowing-up a nodal singularity

5.1.1 Blow-up review

We start this section by recalling the blow-up construction. The blow-up of $\mathbb{C}^n$ at 0 is

$$X = \{(x, [u]) \in \mathbb{C}^n \times \mathbb{P}^{n-1} : x \in [u]\},$$

where $x \in [u]$ means $x = \lambda u$ for some $\lambda \in \mathbb{C}$. The exceptional divisor $E \subset X$ is the set of points $(0, [u])$, so that $x = 0$ and E is a copy of $\mathbb{P}^{n-1}$. The projection

$$\pi : X \to \mathbb{C}^n, \quad \pi(x, [u]) = x$$

satisfies $\pi^{-1}(0) = E = \mathbb{P}^{n-1}$ and π is a biholomorphism on $X \backslash E$.

We work out the coordinates for $n = 2$. In this case, X consists of pairs

$$\{(x, y), [u : v]\} \in \mathbb{C}^2 \times \mathbb{P}^1$$

such that

$$\frac{x}{u} = \frac{y}{v}.$$

To obtain coordinate charts on X, we use the coordinate charts on $\mathbb{P}^1$. So in this case, there are two charts:

• Chart $U = \{u = 1\}$. Coordinates are defined by

$$\begin{aligned} x &= x \\ y &= xv \\ u &= 1 \\ v &= v, \end{aligned}$$

so that we only keep the two coordinate (x, v) on this patch on X. The exceptional divisor (where $(x, y) = (0, 0)$) appears as $E \cap U = \{x = 0\}$, and v is a free coordinate.

• Chart $V = \{v = 1\}$. Coordinates are defined by

$$\begin{aligned} x &= uy \\ y &= y \\ u &= u \\ v &= 1, \end{aligned}$$

and we only keep the coordinates (y, u) on this set. The exceptional divisor is $E \cap V = \{y = 0\}$ with free coordinate u.

On $U \cap V$, the change of coordinates from (x, v) to (y, u) on the overlap is

$$x = uy, \quad v = u^{-1}.$$

Indeed, in chart U, we have $v = y/x$ which substituting in chart V givs $v = y/uy = u^{-1}$. The change of coordinates $v = u^{-1}$ is the coordinate change on $\mathbb{P}^1$. The change $x = uy$ is the coordinate change on a line bundle over $\mathbb{P}^1$. Recall in general a section over a line bundle over $\mathbb{P}^1$ transforms as

$$\tilde{\sigma}(v) = \tau(u)\sigma(u)$$

where $\tau(u)$ is a local function on $\mathbb{P}^1$ which is the transition function of the bundle. In the case above, we have $\tau(u) = u$. This transition function defines a line bundle

$$\mathcal{O}(-1) \to \mathbb{P}^1.$$

So X is the total space of this line bundle. The other way to define $\mathcal{O}(-1)$ is to cover $\mathbb{P}^1 = \{[u_1 : u_2]\}$ by $U_1 = \{u_1 \neq 0\}$ and $U_2 = \{u_2 \neq 0\}$. Then declare the transition function on the overlap $U_1 \cap U_2$ to be

$$\tau_{12} = \frac{u_1}{u_2},$$

so that local sections transform as $\sigma_1 = \tau_{12}\sigma_2$. Coordinates over U_1 are $\zeta = u_2/u_1$ and coordinates over U_2 are $\xi = u_1/u_2$. Then $\xi = \zeta^{-1}$ and over U_2, we do have $\tau_{12}(\xi) = \xi$. Thus this is the same space as above.

5.1.2 ODP in $\mathbb{C}^3$

We now illustrate how the blow-up procedure can be used to resolve singularities. Consider the space

$$X = \{xz - y^2 = 0\} \subset \mathbb{C}^3.$$

This space is of the form $F(x, y, z) = 0$ with $DF = (z, -2y, x)$, and there is a singularity preventing it from being a submanifold at $(0, 0, 0)$. We can resolve this singularity

$$\sigma : \tilde{X} \to X$$

by blowing up the origin. That is, we look at pairs

$$\{(x, y, z), [u : v : w]\} \in \mathbb{C}^3 \times \mathbb{P}^2$$

with the relation

$$\frac{x}{u} = \frac{y}{v} = \frac{z}{w}.$$

The $\mathbb{P}^2$ will produce 3 coordinate charts that we now describe:

- Chart $U = \{u = 1\}$. Coordinates (x, v, w) satisfy:

$$x = x, \quad y = vx, \quad z = wx.$$

and $u = 1$, $v = v$, $w = w$. The defining equation for X becomes

$$x(wx) - (vx)^2 = 0$$

which is $x^2(w - v^2) = 0$. Recall $E \cap U = \{x = 0\}$ is the exceptional divisor, so we throw out the x^2. The resolution of X is in this chart is then defined by

$$w - v^2 = 0$$

which is smooth, and called the proper transform. Therefore $\tilde{X}$ has two coordinates (x, v), and the exceptional divisor in $\mathbb{X}$ is the curve $x = 0$ with v a free coordinate.

• Chart $V = \{v = 1\}$. Coordinates (y, u, w) satisfy

$$x = uy, \quad y = y, \quad z = wy,$$

and the proper transform is

$$uw - 1 = 0.$$

If $g = uw - 1$, the only problem with $Dg = (w, u)$ is when $(w, u) = 0$ which is not on the curve $uw - 1 = 0$. So $\tilde{X}$ is smooth with two coordinates (y, u) and exceptional divisor at $y = 0$.

• Chart $W = \{w = 1\}$. Coordinates (z, u, v) satisfy:

$$x = uz, \quad y = vz, \quad z = z.$$

and $u = u$, $v = v$, $w = 1$. The proper transform is

$$u - v^2 = 0.$$

Therefore $\tilde{X}$ has two coordinates (z, v), and the exceptional divisor is the curve $z = 0$ with v a free coordinate.

Next, we note that $\tilde{X}$ is the total space of the line bundle $\mathcal{O}(-2) \to \mathbb{P}^1$. To see the $\mathbb{P}^1$ coordinate, convert from U to V on $U \cap V$ to obtain

$$v = y/x = y/uy = 1/u.$$

For the fiber,

$$x = y$$

on $U \cap V$, so nothing going on here. Converting U to W on $U \cap W$ gives $v = v$, but for the fiber

$$x = uz = v^2 z.$$

which is the transition function for $\mathcal{O}(-2)$. Recall that by definition sections of $\mathcal{O}(-2)$ transform as

$$\tilde{\sigma} = \tau^2 \sigma$$

where $\tau(v) = v$ is the transition function on $\mathcal{O}(-1)$.

5.1.3 ODP in $\mathbb{C}^4$

Before discussing singularities, we recall that the blow-up of $\mathbb{C}^n$ along a subspace Z replaces Z by the projectivization of its normal bundle $\mathbb{P}(NZ)$. We give the concrete example of the blow-up of

$$Z = \{z_2 = z_4 = 0\} \subset \mathbb{C}^4.$$

The blow-up along Z is

$$X = \{((z_1, z_2, z_3, z_4), [u : v]) \in \mathbb{C}^4 \times \mathbb{P}^1 : (z_2, z_4) \in [u : v]\},$$

in other words
$$\frac{z_2}{u} = \frac{z_4}{v}.$$

The $\mathbb{P}^1$ provides two charts for Z.

• Chart $U = \{u = 1\}$. Coordinates (z_1, z_2, z_3, v) are defined by
$$z_1 = z_1, \quad z_2 = z_2, \quad z_3 = z_3, \quad z_4 = vz_2, \quad u = 1, \quad v = v$$
with exceptional divisor $E \cap U = \{z_1 = z_2 = z_3 = 0\}$.

• Chart $W = \{v = 1\}$. Coordinates (z_1, z_3, z_4, u) are defined by
$$z_1 = z_1, \quad z_2 = uz_4, \quad z_3 = z_3, \quad z_4 = z_4, \quad u = u, \quad v = 1$$
with exceptional divisor $E \cap V = \{z_1 = z_3 = z_4 = 0\}$.

Consider now the ODP singularity
$$V = \{z_1 z_2 - z_3 z_4 = 0\} \subset \mathbb{C}^4.$$
Note that this singulariy is the same as $\{\sum_i z_i^2 = 0\}$ by a change of coordinates (4.24). We can desingularize this space by blowing-up $\mathbb{C}^4$ along $Z = \{z_2 = z_4 = 0\}$ and taking the proper transform $\tilde{V}$.

• Chart $U = \{u = 1\}$. Using the relations above, the equation for V becomes
$$z_1 z_2 - z_3(vz_2) = 0.$$
We throw out $z_2 = 0$ to obtain the proper transform
$$z_1 - vz_3 = 0,$$
which is now smooth. Therefore here $\tilde{V}$ has three coordinates (z_2, z_3, v), and the relations are
$$z_1 = vz_3, \quad z_2 = z_2, \quad z_3 = z_3, \quad z_4 = vz_2$$
on U.

• Chart $W = \{v = 1\}$. Similarly as above, the proper transform appears as
$$uz_1 - z_3 = 0,$$
and here $\tilde{V}$ has three coordinates (z_1, z_4, u) and the relations are
$$z_1 = z_1, \quad z_2 = uz_4, \quad z_3 = uz_1, \quad z_4 = z_4.$$
on W.

We note that $\tilde{V}$ can be identified with the total space of $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$. For this, we compute the change of coordinates on the overlap $U \cap W$. Relabel
$$(\sigma, s, \zeta) = (z_2, z_3, v), \quad \text{on } U$$

$$(\tilde{s}, \tilde{\sigma}, \xi) = (z_1, z_4, u) \quad \text{on } W.$$

Then chasing relations gives for example,

$$\tilde{s} = z_1 = vz_3 = \zeta s,$$

and altogether

$$\tilde{s} = \zeta s, \quad \tilde{\sigma} = \zeta\sigma, \quad \xi = \zeta^{-1}.$$

This is the change of coordinates on the total space of $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$ with fiber coordiantes s, σ and $\mathbb{P}^1$ coordinate ζ. Thus we have a resolution of singularities

$$\sigma : \tilde{V} \to V$$

where the origin in V is replaced by $\sigma^{-1}(0) = \mathbb{P}^1$. This can be seen since in local coordinates on U then $\sigma^{-1}(0)$ is freely parametrized by coordinate v (and coordinates $z_2 = z_3 = 0$), and on V then σ^{-1} is freely parametrized by coordinate u (and coordinates $z_1 = z_4 = 0$), and on the overlap we noted $u = v^{-1}$. We note that the modulus

$$\|z\|^2 = \sum_{i=1}^{4} |z_i|^2$$

on $V \subset \mathbb{C}^4$ becomes on $\tilde{V}$ the function

$$\|z\|^2 = |vz_3|^2 + |z_2|^2 + |z_3|^2 + |vz_2|^2 = (1 + |\zeta|^2)(|\sigma|^2 + |s|^2)$$

over the U chart. In the V chart, then

$$\|z\|^2 = (1 + |\xi|^2)(|\tilde{\sigma}|^2 + |\tilde{s}|^2).$$

We recognize this as

$$\|z\|^2 = |(\sigma, \zeta)|^2_{h_{FS}}$$

where h_{FS} is the Fubini-Study metric on the $\mathcal{O}(-1)$ fibers.

In summary, the total space of the bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$ is given by two trivializations $\{U, (\sigma, s, \zeta)\}$ and $\{V, (\tilde{\sigma}, \tilde{s}, \xi)\}$ with $\xi = 1/\zeta$ and $\tilde{s} = \zeta s$, $\tilde{\sigma} = \zeta\sigma$. The set $\|z\| = 0$ is a $\mathbb{P}^1$.

Definition 5.1. *A $(-1,-1)$ curve C in a compact threefold X is defined by $C \cong \mathbb{P}^1$ and there exists a neighborhood of C which is biholomorphic to a neighborhood of $\{\|z\| = 0\}$ in the total space $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$.*

5.2 Smoothing a nodal singularity

We return to the nodal singularity

$$V = \{\sum_{i=1}^{4} (z_i)^2 = 0\} \subseteq \mathbb{C}^4.$$

We see that V is a cone, since if $z \in V$ then so is λz for $\lambda \in \mathbb{C}$. We also have the radius function

$$\|z\|^2 = \sum_i |z_i|^2.$$

On this cone, we have the holomorphic volume form

$$\Omega = \frac{1}{z_1} dz_2 \wedge dz_3 \wedge dz_4, \quad \text{on } \{z_1 \neq 0\}$$

and the corresponding formula on the other open sets $\{z_i \neq 0\}$. These local expressions glue on V to give a global holomorphic volume form. Next, we write

$$z_k = x_k + iy_k.$$

The equation becomes

$$0 = (|x|^2 - |y|^2) + 2i\langle x, y\rangle.$$

Since $\|z\|^2 = |x|^2 + |y|^2$, this constraint is equivalent to

$$|x|^2 = \frac{\|z\|^2}{2}, \quad |y|^2 = \frac{\|z\|^2}{2}, \quad \langle x, y\rangle = 0.$$

For each fixed $r > 0$, the cross-section where $\|z\| = r$ is seen from here to be a S^2 bundle over S^3. These are topologically trivial, so for each $r > 0$, the cross section where $\|z\| = r$ is $S^3 \times S^2$.

The smoothing of V is given by

$$V_t = \{\sum_{i=1}^{4} (z_i)^2 = t\}.$$

This is smooth because the only point where the derivative of $\sum_{i=1}^{4} (z_i)^2$ is not surjective is at the origin, which is not included in V_t.

After rotating coordinates z, we may suppose $t > 0$. The defining equation becomes

$$t = |x|^2 - |y|^2, \quad \langle x, y\rangle = 0.$$

Since $\|z\|^2 = t + 2|y|^2$, we see that

$$\|z\|^2 \geqslant t.$$

The point $\{\|z\|^2 = 0\} \subset V$ has been inflated to $\{\|z\|^2 = t\} \subset V_t$, which is

$$|x|^2 = t, \quad y = 0.$$

The space of x describes an S^3 of vanishing radius as $t \to 0$. This is sometimes called the vanishing sphere.

Lastly, we note that V_t also admits a holomorphic volume form, given by local expressions such as

$$\Omega_t = \frac{1}{z_1} dz_2 \wedge dz_3 \wedge dz_4, \quad \text{on } \{z_1 \neq 0\}.$$

5.3 Candelas-de la Ossa metrics

To summarize our discussion so far, we have described two ways to resolve the singular space

$$V = \{\sum z_i^2 = 0\} \subset \mathbb{C}^4$$

The first is by small blow-up, which replaced the origin by $\mathbb{P}^1 = S^2$. We call this the small resolution

$$\sigma : \tilde{V} \to V.$$

The second is by smoothing, which replaced the origin by S^3.

$$V_t = \{\sum z_i^2 = t\} \subset \mathbb{C}^4.$$

Candelas-de la Ossa [2] constructed a sequence of metrics on both sides. We will discuss how one side of the sequence sends the area of a holomorphic curve $\mathbb{P}^1$ to zero, and the other side sends the area of a special Lagrangian 3-sphere to zero.

5.3.1 Metrics on the small resolution

We work on the total space $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$ with two coordinate charts (λ, u, v) and $(\tilde{\lambda}, \tilde{u}, \tilde{v})$ satisfying the change of coordinates

$$\lambda = \tilde{\lambda}^{-1}, \quad u = \tilde{\lambda}\tilde{u}, \quad v = \tilde{\lambda}\tilde{v}.$$

We have the well-defined function

$$\tau = (1 + |\lambda|^2)(|u|^2 + |v|^2),$$

which measures the distance to the zero section $\mathbb{P}^1$ (coordinate λ) along the $\mathcal{O}(-1)$ fibers (coordinates u, v) using the Fubini-Study metric. For $a > 0$, the Candelas-de la Ossa ansatz is

$$\omega_{co,a} = i\partial\bar{\partial} f_a(\tau) + 4a^2 \omega_{FS}.$$

for some function f_a. Note that

$$\int_{\mathbb{P}^1} \omega_{co,a} = 4a^2 \mathrm{Vol}(\mathbb{P}^1, \omega_{FS}) \to 0, \quad a \to 0.$$

We want to solve for f_a such that these metrics are Ricci-flat. We expand the ansatz as

$$i\partial\Big[f'\tau\bar{\partial}\log\tau\Big] + 4a^2\omega_{FS}$$

and using $\omega_{FS} = i\partial\bar{\partial}\log(1 + |\lambda|^2)$,

$$\begin{aligned} \omega_{co,a} &= \Big[(f'' + \tau^{-1}f')i\partial\tau \wedge \bar{\partial}\tau\Big] + \Big[f'\tau i\partial\bar{\partial}\log(|u|^2 + |v|^2)\Big] \\ &\quad + \Big[(f'\tau + 4a^2)\omega_{FS}\Big]. \end{aligned} \tag{5.1}$$

To compute the Ricci curvature, we first compute $\omega_{co,a}^3$. We note that these 3 terms each square to zero. For example

$$\left[i\partial\bar{\partial}\log(|u|^2+|v|^2)\right]^2=0. \tag{5.2}$$

Indeed,

$$\partial\bar{\partial}\log(|u|^2+|v|^2)=\frac{1}{(|u|^2+|v|^2)^2}\left\{\left[(|u|^2+|v|^2)(du\wedge d\bar{u}+dv\wedge d\bar{v})\right]-\left[(\bar{u}du+\bar{v}dv)\wedge(ud\bar{u}+vd\bar{v})\right]\right\}.$$

Squaring this is of the form $(\alpha-\beta)^2$ where $\beta^2=0$. A direct computation then shows $\alpha^2-2\alpha\beta=2(|u|^2+|v|^2)^2-2(|u|^2+|v|^2)^2=0$. This verifies (5.2).

Going back to (5.1), computing $\omega_{co,a}^3$ is of the form $(a+b+c)^3=6abc$ since $a^2=b^2=c^2=0$. We have

$$\omega_{co,a}^3=6(f''+\tau^{-1}f')(f'\tau)(f'\tau+4a^2)i\partial\tau\wedge\bar{\partial}\tau\wedge\omega_{FS}\wedge i\partial\bar{\partial}\log(|u|^2+|v|^2).$$

Next, we have

$$\omega_{FS}=(1+|\lambda|^2)^{-2}id\lambda\wedge d\bar{\lambda},$$

and

$$\partial\tau\wedge\bar{\partial}\tau=\Big((|u|^2+|v|^2)\bar{\lambda}d\lambda+(1+|\lambda|^2)(\bar{u}du+\bar{v}dv)\Big)\Big((|u|^2+|v|^2)\lambda d\bar{\lambda}+(1+|\lambda|^2)(ud\bar{u}+vd\bar{v})\Big)$$

and

$$\partial\bar{\partial}\log(|u|^2+|v|^2)=\frac{1}{(|u|^2+|v|^2)^2}\Big((|u|^2+|v|^2)(du\wedge d\bar{u}+dv\wedge d\bar{v}-(\bar{u}du+\bar{v}dv)\wedge(ud\bar{u}+vd\bar{v})\Big)$$

Using this, direct calculation gives

$$i\partial\tau\wedge\bar{\partial}\tau\wedge\omega_{FS}\wedge i\partial\bar{\partial}\log(|u|^2+|v|^2)=c(id\lambda\wedge d\bar{\lambda})\wedge(idu\wedge d\bar{u})\wedge(idv\wedge d\bar{v}).$$

Therefore, writing $\omega_{co,a}=i(g_{co,a})_{j\bar{k}}dz^j\wedge d\bar{z}^k$, then up to a constant we have

$$\det g_{co,a}=(f''f'\tau+(f')^2)(f'\tau+4a^2).$$

Side note: at $\lambda=0$, in (λ,u,v) coordinates the metric is

$$g_{co,a}(0,u,v)=\begin{bmatrix} f'\tau+4a^2 & 0 & 0\\ 0 & f'+|u|^2f'' & f''u\bar{v}\\ 0 & f''v\bar{u} & f'+|v|^2f''\end{bmatrix}.$$

Let $\gamma(\tau)=\tau f'$. The determinant in terms of γ is

$$\det g_{co,a}=\gamma'\gamma(\gamma+4a^2)\tau^{-1}$$

To find a Ricci-flat metric, from $R_{j\bar{k}}=-\partial_j\partial_{\bar{k}}\log\det g$ we look for solutions to $\det g_{co,a}=const$, which is the equation (with convenient choice of constant)

$$\gamma'\gamma(\gamma+4a^2)=\frac{2}{3}\tau.$$

This has solution

$$\gamma^3 + 6a^2\gamma^2 = \tau^2.$$

This is

$$\tau(f_a')^3 + 6a^2(f_a')^2 = 1.$$

From here, we note that if f_1 is a solution for $a = 1$, then $f_a = a^2 f_1(\tau/a^3)$ is a solution for arbitrary a. We discuss the solution f for $a = 1$ and obtain the other f_a by rescaling.

We look for a positive solution of $\tau y^3 + 6y^2 = 1$ with $y = f'(\tau)$. At $\tau = 0$, we choose the solution $y = 1/\sqrt{6}$. It turns out that there is an explicit solution $y(\tau) > 0$ on $\tau \geqslant 0$ given by

$$y = \frac{1}{\tau}\left[z + \frac{4}{z} - 2\right], \quad z = 2^{-1/3}(\tau^2 + (\tau^4 - 32\tau^2)^{1/2} - 16)^{1/3}. \tag{5.3}$$

The function f_1 is then defined on $[0,\infty)$ and given by

$$f_1(\tau) = \int_0^\tau y(t)dt.$$

By construction, it solves

$$(f''f'\tau + (f')^2)(f'\tau + 4) = 1.$$

The form $\omega_{co,1}$ is > 0, since it is given by (5.1) with $f' > 0$ and $f''\tau + f' > 0$.

Next, we show that $f_a \to f_0$ smoothly on compact sets away from $u = v = 0$. We study the asymptotic behavior of f_1 as $\tau \to \infty$. For $\tau \geqslant K$, then the Puiseux series of (5.3) turns out to be

$$y = \tau^{-1/3} - \frac{2}{\tau} + 4\tau^{-5/3} - \frac{16}{3}\tau^{-7/3} + \dots$$

Integrating gives

$$f_1(\tau) = c_0\tau^{2/3} + c_1 \log\tau + c_2\tau^{-2/3} + c_3\tau^{-4/3} + \dots$$

Therefore when $\tau \geqslant Ka^3$, then

$$f_a(\tau) = c_0\tau^{2/3} + c_1 a^2 \log a^{-3}\tau + c_2 a^4\tau^{-2/3} + c_3 a^6 \tau^{-4/3} + \dots.$$

Therefore $f_a \to c_0 f_0$ as $a \to 0$ on compact sets disjoint from $\{\tau = 0\}$, where

$$f_0 = \tau^{2/3}, \quad \omega_{co,0} = i\partial\bar\partial\tau^{2/3}.$$

At the level of metrics, the expansion is

$$\omega_{co,a} - \omega_{co,0} = 4a^2\omega_{FS} - c_1 a^2 i\partial\bar\partial \log r - c_2 a^4 i\partial\bar\partial r^{-2} - c_3 a^6 i\partial\bar\partial r^{-4} + \dots$$

on $\{r \geqslant K_0 a\}$, where $r = \tau^{1/3}$. Therefore $\omega_{co,a} \to \omega_{co,0}$ as $a \to 0$ uniformly on compact sets which are disjoint from the exceptional curve $\mathbb{P}^1 = \{\tau = 0\}$.

Remark 5.2. Let us rewrite things slightly differently. Rescale the metrics so that $c_0 = 1$, let $r = \tau^{1/3}$ so that $r = (1 + |\lambda|^2)^{1/3}(|u|^2 + |v|^2)^{1/3}$ and

$$\omega_{co,0} = i\partial\bar\partial r^2, \quad \omega_{co,a} = a^2 S_{a^{-1}}^* \omega_{co,1}$$

where

$$S_R(\lambda, u, v) = (\lambda, R^{3/2}u, R^{3/2}v)$$

satisfies $S_R^* r = Rr$ and $S_R^* \omega_{co,0} = R^2 \omega_{co,0}$. Let $A_R = \{R \leqslant r \leqslant 2R\}$ and note $S_R : A_1 \to A_R$. We pullback the compact set estimate

$$|i\partial\bar{\partial}(f_1 - f_0)|_{\omega_0} \leqslant C, \quad \{1 \leqslant r \leqslant 2\}$$

to obtain

$$|i\partial\bar{\partial}S_\rho^*(f_1 - f_0)|_{S_\rho^*\omega_0} \leqslant C, \quad \{\rho^{-1} \leqslant r \leqslant 2\rho^{-1}\}$$

which, using $|T|_{S_\rho^*\omega_0} = |T|_{\rho^2\omega_0} = \rho^{-2}|T|_{\omega_0}$ for $T = [T_{ij}]$, implies

$$\sup_{\rho^{-1} \leqslant r \leqslant 2\rho^{-1}} |\omega_{co,1} - \omega_{co,0}|_{\omega_0} \leqslant C\rho^2$$

and

$$|\omega_{co,1} - \omega_0|_{\omega_{co,0}} \leqslant Cr^{-2}.$$

We now pullback this estimate via $S_{a^{-1}}$.

$$|S_{a^{-1}}^*(\omega_{co,1} - \omega_{co,0})|_{S_{a^{-1}}^*\omega_{co,0}} \leqslant Ca^2 r^{-2}.$$

Therefore

$$|S_{a^{-1}}^*\omega_{co,1} - a^{-2}\omega_{co,0}|_{\omega_{co,0}} \leqslant Cr^{-2},$$

and

$$|\omega_{co,a} - \omega_{co,0}|_{\omega_{co,0}} \leqslant Cr^{-2}a^2$$

which shows that $\omega_{co,a} \to \omega_{co,0}$ as $a \to 0$ on sets $\{r > \varepsilon\}$.

5.3.2 Metrics on the smoothings

In this section, we construct Kähler Ricci-flat metrics on V_t. For $t \in \mathbb{C}$, consider as before

$$V_t = \{\sum_{i=1}^4 (w_i)^2 = t\} \subseteq \mathbb{C}^4$$

with radius function $\tau = \sum |w_i|^2$. Consider potentials of the form

$$\varphi(z) = f(\tau),$$

for f to be determined. To find Kähler Ricci-flat metrics, we solve

$$(i\partial\bar{\partial}\varphi)^3 = i\Omega \wedge \bar{\Omega}.$$

We compute

$$\bar{\partial}\varphi = f'\bar{\partial}\tau,$$

and

$$i\partial\bar{\partial}\varphi = f' i\partial\bar{\partial}\tau + f'' i\partial\tau \wedge \bar{\partial}\tau.$$

Its Monge-Ampère mass is

$$(i\partial\bar{\partial}\varphi)^3 = (f')^3(i\partial\bar{\partial}\tau)^3 + 3(f')^2 f''(i\partial\bar{\partial}\tau)^2 \wedge i\partial\tau \wedge \bar{\partial}\tau. \tag{5.4}$$

We have

$$\partial\tau \wedge \bar{\partial}\tau = w_j\bar{w}_i dw_i \wedge d\bar{w}_j, \quad \partial\bar{\partial}\tau = \sum_k dw_k \wedge d\bar{w}_k.$$

We now compute in coordinates (w_1, w_2, w_3) on V_t over the chart $\{w_4 \neq 0\}$. On $\{w_4 \neq 0\}$, we have

$$\Omega_t = \frac{1}{w_4}\, dw_1 \wedge dw_2 \wedge dw_3.$$

Therefore

$$i\Omega_t \wedge \bar{\Omega}_t = \frac{1}{|w_4|^2}(idw_1 \wedge d\bar{w}_1) \wedge (idw_2 \wedge d\bar{w}_2) \wedge (idw_3 \wedge d\bar{w}_3).$$

We write

$$d\mu = \frac{1}{|w_4|^2}(idw_1 \wedge d\bar{w}_1) \wedge (idw_2 \wedge d\bar{w}_2) \wedge (idw_3 \wedge d\bar{w}_3)$$

for simplicity. We need to equate this to $(i\partial\bar{\partial}\varphi)^3$ (5.4). For this we will use the defining relation of V_t, which gives

$$\sum_{k=1}^4 w_k dw_k = 0, \quad dw_4 = -\frac{1}{w_4}(w_1 dw_1 + w_2 dw_2 + w_3 dw_3).$$

Using this, we can compute the first term in (5.4) to be

$$(i\partial\bar{\partial}\tau)^3 = (3!)\tau d\mu.$$

Indeed,

$$\begin{aligned} (i\partial\bar{\partial}\tau)^3 &= (i^3 3!)[dw_{1\bar{1}2\bar{2}3\bar{3}} + dw_{1\bar{1}2\bar{2}4\bar{4}} + dw_{1\bar{1}3\bar{3}4\bar{4}} + dw_{2\bar{2}3\bar{3}4\bar{4}}] \\ &= (i^3 3!)\left[1 + \frac{1}{|w_4|^2}(|w_3|^2 + |w_2|^2 + |w_1|^2)\right] dw_{1\bar{1}2\bar{2}3\bar{3}}. \end{aligned} \tag{5.5}$$

Next we compute the second term in (5.4). We start with

$$\begin{aligned} & (idw_1 \wedge d\bar{w}_1) \wedge (idw_2 \wedge d\bar{w}_2) \wedge i\partial\tau \wedge \bar{\partial}\tau \\ =\ & \lambda_{1\bar{1}} \wedge \lambda_{2\bar{2}} \wedge (|w_3|^2\lambda_{3\bar{3}} + |w_4|^2\lambda_{4\bar{4}} + w_4\bar{w}_3\lambda_{3\bar{4}} + w_3\bar{w}_4\lambda_{4\bar{3}}) \end{aligned} \tag{5.6}$$

with the notation $\lambda_{i\bar{j}} = idz_i \wedge d\bar{z}_j$. By the defining relation

$$\begin{aligned} & \lambda_{1\bar{1}} \wedge \lambda_{2\bar{2}} \wedge i\partial\tau \wedge \bar{\partial}\tau \\ =\ & (|w_3|^2 + |w_3|^2 - \frac{w_4(\bar{w}_3)^2}{\bar{w}_4} - \frac{\bar{w}_4(w_3)^2}{w_4})\lambda_{1\bar{1}} \wedge \lambda_{2\bar{2}} \wedge \lambda_{3\bar{3}} \\ =\ & 2\left(|w_3|^2|w_4|^2 - \mathrm{Re}\,(w_4\bar{w}_3)^2\right) d\mu \end{aligned} \tag{5.7}$$

We also have

$$\begin{aligned} & \lambda_{1\bar{1}} \wedge \lambda_{4\bar{4}} \wedge i\partial\tau \wedge \bar{\partial}\tau \\ =\ & \lambda_{1\bar{1}} \wedge \lambda_{4\bar{4}} \wedge (|w_2|^2\lambda_{2\bar{2}} + |w_3|^2\lambda_{3\bar{3}} + w_3\bar{w}_2\lambda_{2\bar{3}} + w_2\bar{w}_3\lambda_{3\bar{2}}) \end{aligned} \tag{5.8}$$

which by the defining relation becomes

$$\begin{aligned}
& \lambda_{1\bar{1}} \wedge \lambda_{4\bar{4}} \wedge i\partial\tau \wedge \bar{\partial}\tau \\
= \quad & (\frac{|w_2|^2|w_3|^2}{|w_4|^2} + \frac{|w_2|^2|w_3|^2}{|w_4|^2} - \frac{(w_3\bar{w}_2)^2}{|w_4|^2} - \frac{(\bar{w}_3 w_2)^2}{|w_4|^2})\lambda_{1\bar{1}} \wedge \lambda_{2\bar{2}} \wedge \lambda_{3\bar{3}} \\
= \quad & 2(|w_2|^2|w_3|^2 - \mathrm{Re}\,(w_3\bar{w}_2)^2)d\mu.
\end{aligned} \tag{5.9}$$

By symmetry,

$$\begin{aligned}
& (i\partial\bar{\partial}\tau)^2 \wedge i\partial\tau \wedge \bar{\partial}\tau \\
= \quad & 4\bigg(|w_1|^2|w_4|^2 + |w_2|^2|w_4|^2 + |w_3|^2|w_4|^2 + |w_2|^2|w_3|^2 + |w_3|^2|w_1|^2 + |w_2|^2|w_1|^2 \\
& -\mathrm{Re}\,\{(w_1\bar{w}_2)^2 + (w_1\bar{w}_3)^2 + (w_1\bar{w}_4)^2 + (w_2\bar{w}_3)^2 + (w_2\bar{w}_4)^2 + (w_3\bar{w}_4)^2\}\bigg)d\mu
\end{aligned}$$

Since $|t|^2 = |\sum w_i^2|^2$ and $\tau^2 = (\sum |w_i|^2)^2$, we obtain

$$(i\partial\bar{\partial}\tau)^2 \wedge i\partial\tau \wedge \bar{\partial}\tau = 2(\tau^2 - |t|^2)d\mu.$$

Substituting this into (5.4) gives

$$(i\partial\bar{\partial}\varphi)^3 = 3!\bigg[(f')^3\tau + (f')^2 f''(\tau^2 - |t|^2)\bigg]d\mu.$$

Thus to solve $(i\partial\bar{\partial}\varphi)^3 = i\Omega \wedge \overline{\Omega}$, we need to solve

$$(f')^3\tau + (f')^2 f''(\tau^2 - |t|^2) = 1/6,$$

for a function $f(\tau)$, and where t is a fixed parameter. Here is the result. When $t = 0$, the solution is proportional to

$$f_0(\tau) = \tau^{2/3}$$

and for given $t \neq 0$, the solution is proportional to

$$f_t(\tau) = 2^{-1/3}|t|^{2/3}\int_0^{\cosh^{-1}(\tau/|t|)} (\sinh(2\lambda) - 2\lambda)^{1/3} d\lambda.$$

Remark 5.3. Let's work out how to find this solution when $t = 0$. The ODE is

$$6\tau[(f')^3 + \tau(f')^2 f''] = 1.$$

Let $\tau = s^2$ and $\gamma(s) = s^2 f'(s^2)$. Then we have

$$(\gamma^3)' = (s^6(f')^3)' = 6s^5 f'^3 + (3)s^6 f'^2(2s)f'' = 6s^2[f'^3 + s^2 f'^2 f'']s^3.$$

The equation becomes

$$(\gamma^3)' = s^3,$$

which admits the solution $\gamma^3 = \frac{1}{4}s^4$. Therefore

$$s^6[f'(s^2)]^3 = \frac{1}{4}s^4$$

which for $f' > 0$ is the ODE

$$f'(\tau) = c_0\tau^{-1/3}.$$

The solution is $f(\tau) = c_1\tau^{2/3}$.

Remark 5.4. Let's compute the asymptotics of f_1. Its derivative is

$$f_1'(\tau) = \left(\sinh(2\cosh^{-1}(\tau)) - 2\cosh^{-1}\tau\right)^{1/3}\frac{1}{\sqrt{\tau^2-1}}.$$

We have $\sinh(2\cosh^{-1}(\tau)) = 2\tau(\tau-1)^{1/2}(\tau+1)^{1/2}$ and $\cosh^{-1}\tau = \log(\tau+\sqrt{\tau^2-1})$. Then as $\tau \to \infty$, the series expansion turns out to be

$$f'(\tau) = 2^{1/3}\tau^{-1/3} + a_1\tau^{-7/3}\log\tau + a_2\tau^{-7/3} + O(\tau^{-10/3})$$

Integrating gives

$$f(\tau) = c_0\tau^{2/3} + a_1\tau^{-4/3}\log\tau + a_2\tau^{-4/3} + O(\tau^{-7/3}).$$

Therefore, if we let $r = \tau^{1/3}$, then

$$\varphi_1 = c_0r^2 + a_1r^{-4}\log r + a_2r^{-4} + O(r^{-7}).$$

Remark 5.5. The metrics $\omega_{co,1}$ are asymptotically conical. We now describe what this means (see [6] for more on asymptotically conical Kähler-Ricci flat metrics). Let the Kähler-Ricci flat metrics be denoted

$$\omega_{co,0} = i\partial\bar{\partial}\varphi_0, \quad \omega_{co,1} = i\partial\bar{\partial}\varphi_1.$$

We consider the map $\Phi : V \cap \{|z| > R\} \to V_1$ given by

$$\Phi(x) = x + \frac{\bar{x}}{2|x|^2}.$$

We will estimate the order of $\Phi^*\omega_{co,1} - \omega_{co,0}$. By the asymptotics of φ_1 and $r = \|x\|^{2/3}$,

$$\begin{aligned}(\Phi^*\varphi_1)(x) &= |\Phi(x)|^{4/3} + \mathcal{O}(|\Phi(x)|^{-(8/3)}\log|\Phi(x)|) \\ &= |x|^{4/3} + \mathcal{O}(|x|^{(4/3)-1}|x|^{-1}) + \mathcal{O}(|\Phi(x)|^{-(8/3)}\log|\Phi(x)|) \\ &= r^2 + \mathcal{O}(r^{-1})\end{aligned} \tag{5.10}$$

Therefore

$$\Phi^*\varphi_1 - \varphi_0 = \mathcal{O}(r^{-1}).$$

Next, we start from

$$\sup_{1\leqslant r\leqslant 2}|\Phi^*\omega_{co,1} - \omega_{co,0}|_{\omega_{co,0}} \leqslant C,$$

and pull this back via $S_\lambda : V \to V$, $S_\lambda(z) = \lambda^{3/2} z$ to get

$$\sup_{\lambda^{-1} \leqslant r \leqslant 2\lambda^{-1}} |S_\lambda^* i\partial\bar{\partial}(\Phi^* \varphi_1 - \varphi_0)|_{S_\lambda^* \omega_{co,0}} \leqslant C.$$

The definition of S_λ is such that $S_\lambda^* r = \lambda r$ and $S_\lambda^* \omega_{co,0} = \lambda^2 \omega_{co,0}$. Since

$$|S_\lambda^* i\partial\bar{\partial}(\Phi^* \varphi_1 - \varphi_0)|_{S_\lambda^* \omega_{co,0}} = \lambda^{-2}\lambda^{-1} |i\partial\bar{\partial}(\Phi^* \varphi_1 - \varphi_0)|_{\omega_{co,0}}$$

we conclude

$$|\Phi^* \omega_{co,1} - \omega_{co,0}|_{\omega_{co,0}} \leqslant C r^{-3}. \tag{5.11}$$

Metrics satisfying estimates of this form are said to be asymptotically conical (with rate 3).

Remark 5.6. Next, we notice that $S_\lambda(z) = \lambda^{3/2} z$ implies

$$S_{t^{1/3}} : V_1 \to V_t.$$

We assume $t > 0$. We have that

$$\omega_{co,t} = t^{2/3} S_{t^{-1/3}}^* \omega_{co,1}$$

is a natural sequence of metrics, and we can check that it does agree with the sequence of explicit metrics obtained earlier. The reason for the $t^{2/3}$ out-front is that far out on the cone $r \gg R$, then $\omega_{co,1} \sim i\partial\bar{\partial} r^2$ and $S_{t^{-1/3}}^* i\partial\bar{\partial} r^2 = t^{-2/3}$, so that these metrics $\omega_{co,t}$ are asymptotic to the cone $i\partial\bar{\partial} r^2$. More precisely, let

$$\Phi_t = S_{t^{1/3}} \circ \Phi \circ S_{t^{-1/3}} : V \to V_t,$$

then

$$|\Phi_t^* \omega_{co,t} - \omega_{co,0}|_{\omega_{co,0}} \leqslant C|t| r^{-3} \tag{5.12}$$

Indeed, pulling back (5.11) gives

$$|S_{t^{-1/3}}^* \Phi^* \omega_{co,1} - S_{t^{-1/3}}^* \omega_{co,0}|_{S_{t^{-1/3}}^* \omega_{co,0}} \leqslant C|t| r^{-3}$$

which combined with

$$|t^{-2/3} \Phi_t^* \omega_{co,t} - t^{-2/3} \omega_{co,0}|_{t^{-2/3}\omega_{co,0}} = |\Phi_t^* \omega_{co,t} - \omega_{co,0}|_{\omega_{co,0}}$$

gives the estimate (5.12). This estimate can be interpreted as $\omega_{co,t} \to \omega_{co,0}$ on compact sets away from the cone singularity as $t \to 0$.

5.3.3 The cone metric

We return to the cone

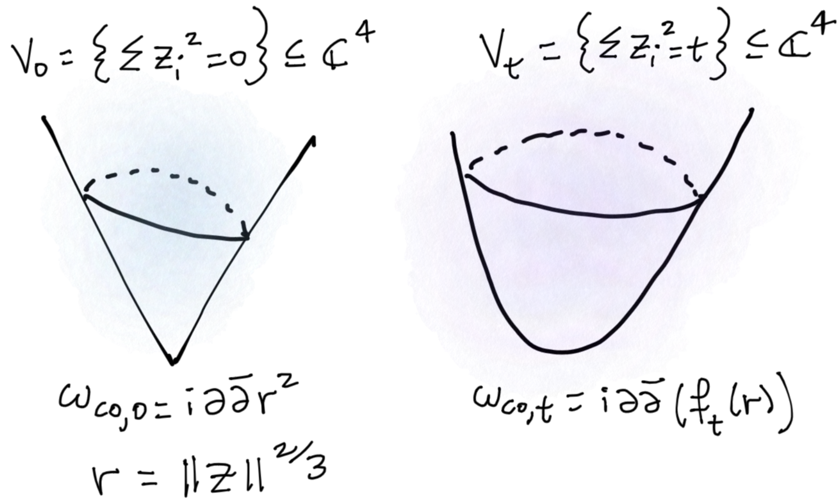
$$V = \{\sum_{i=1}^4 z_i^2 = 0\} \subseteq \mathbb{C}^4$$

with holomorphic volume form $\Omega = \frac{1}{z_1} dz_2 \wedge dz_3 \wedge dz_4$. We will now equip this with a natural Calabi-Yau metric.

Both sequence of metrics $g_{co,a}$ and $g_{co,t}$ agree at $g_{co,0}$. Recall that $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$ with zero section removed can be identified with V and the function τ in both previous sections agrees in this identification. So we will work on V, and the Candelas-de la Ossa cone metric derived in the two previous sections is, up to a constant,

$$\omega_{co,0} = i\partial\bar{\partial}r^2$$

where $r = \|z\|^{2/3}$.



Here is the reason why the power $2/3$ is natural. We are looking to solve $\omega_{co,0}^3 = i\Omega \wedge \bar{\Omega}$. We can rescale $z \mapsto \lambda z$ on both sides, which gives

$$(\lambda^{4/3})^3 = (\lambda^{-1}\lambda^3)(\lambda^{-1}\lambda^3)$$

which is consistent.

Since the cone radius is r, the natural scaling on the cone is

$$t \cdot (z_1, z_2, z_3, z_4) = (t^{3/2}z_1, t^{3/2}z_2, t^{3/2}z_3, t^{3/2}z_4)$$

since $r(t \cdot z) = tr$.

We have explained why this metric is Kähler-Ricci flat. We will now discuss why this is a cone metric. Note that V is a cone, since if $z \in V$ then $\lambda z \in V$ for any $\lambda \in \mathbb{C}$. In general, a cone metric is a metric on $(0, \infty) \times \Sigma$ of the form

$$g = dr \otimes dr + r^2 g_\Sigma. \tag{5.13}$$

This type of metric is very useful in Riemann geometry. To see why, convert the Euclidean metric g_{Euc} on $\mathbb{R}^n$ to polar coordinates (r, θ^i). It will be of the form (5.13) with g_Σ the metric on the sphere S^{n-1}. Riemannian geometry with metrics of the form (5.13) behaves like polar coordinates: for example, the kernel of the Laplacian Δ_g can be decomposed by separation of variables.

Here is why $\omega_{co,0}$ can be put in this form. We insert a factor of $(1/2)$ in the definition of $\omega_{co} = \frac{1}{2}i\partial\bar{\partial}r^2$ for convenience. Compute

$$\omega_{co} = i\partial(r^2\bar{\partial}\log r) = 2i\partial r \wedge \bar{\partial}r + r^2 i\partial\bar{\partial}\log r.$$

We can write this as

$$\omega_{co} = -dr \wedge Jdr + r^2\bigg(- \frac{i}{2} dJd\log r\bigg)$$

where $(J\beta)(X) = \beta(JX)$. This can be seen by writing $dr = \partial r + \bar{\partial} r$ and $Jdr = i\partial r - i\bar{\partial} r$.

Let $\eta = -Jd\log r$. To obtain the metric, we use $g_{co}(X,Y) = \omega_{co}(X, JY)$.

$$g_{co} = dr \otimes dr + r^2 \eta \otimes \eta + r^2 d\eta(\cdot, J\cdot).$$

It remains to show that

$$\eta = \eta_i(\theta) d\theta^i, \quad d\eta(\cdot, J\cdot) = \alpha_{ij}(\theta) d\theta^i \otimes d\theta^j \tag{5.14}$$

where the implicit function theorem gives local coordinates (r, θ^i) near a given point.

• First, we note the following nowhere vanishing holomorphic vector field on V:

$$\xi = 3 \sum_{i=1}^4 z_i \frac{\partial}{\partial z_i} = r\frac{\partial}{\partial r} - iJ\bigg[r\frac{\partial}{\partial r}\bigg],$$

with $r\partial_r = \frac{3}{2} x_i \frac{\partial}{\partial x_i} + \frac{3}{2} y_i \frac{\partial}{\partial y_i}$. This vector field generates the $t\cdot$ action on V in the sense

$$\xi(f) = 2\frac{d}{dt}\bigg|_{t=1} f(t\cdot z, \bar{z}).$$

From here, it follows that

$$\xi(\log r) = 1,$$

so taking real and imaginary parts gives

$$r\frac{\partial}{\partial r}(\log r) = 1, \quad J\bigg[r\frac{\partial}{\partial r}\bigg](\log r) = 0.$$

In $\mathbb{R}^2$ with polar coordinates $re^{i\theta}$, we can think of $J(r\partial_r) = r\partial_\theta$. This discussion implies $\eta(\partial_r) = -J(\partial_r)\log r = 0$ and so

$$\eta = \eta_i(r,\theta) d\theta^i.$$

The next step is to show that η_i does not depend on r, and for this we will use the Lie derivative.

• Next, we note

$$L_\xi J = 0.$$

The Lie derivative is

$$L_\xi(JX) = (L_\xi J)(X) + JL_\xi(X)$$

and $L_\xi(X) = [\xi, X]$ for vector fields X. So this amounts to showing

$$[\xi, J\partial_\alpha] = J[\xi, \partial_\alpha], \tag{5.15}$$

for $\partial_\alpha = \frac{\partial}{\partial z_k}$ or $\partial_\alpha = \frac{\partial}{\partial \bar{z}_k}$. Since $\xi = 3z_i\frac{\partial}{\partial z_i}$, this is readily verified directly.

- We now show that

$$L_{r\partial_r}\eta = 0.$$

By Cartan's formula, $L_{r\partial_r} = d\iota_{r\partial_r}\eta + \iota_{r\partial_r}d\eta$. Since $\iota_{r\partial_r}\eta = 0$, we need to show

$$d\eta(r\partial_r, \cdot) = 0. \tag{5.16}$$

Indeed, the invariant formula for $d\eta$ gives

$$\begin{aligned} d\eta(r\partial_r, X) &= r\partial_r\eta(X) - X\eta(r\partial_r) - \eta([r\partial_r, X]) \\ &= -r\partial_r JX \log r + J[r\partial_r, X]\log r \\ &= -r\partial_r JX \log r + [r\partial_r, JX]\log r \\ &= -JXr\partial_r \log r = 0. \end{aligned} \tag{5.17}$$

Here we used the real part of $[\xi, JX] = J[\xi, X]$ (5.15).

- Altogether, since

$$0 = L_{r\partial_r}\eta = \frac{d}{dt}\bigg|_{t=1}\eta_i(tr,\theta)d\theta^i$$

it follows that in a local chart we can write $\eta = \eta_i(\theta)d\theta^i$ and η_i does not depend on r. In other words, $\eta = p^*\eta_\Sigma$ where $p : V \to \Sigma$ is $p(r,\theta) = \theta$. Similarly, (5.16) implies

$$d\eta(\cdot, J\cdot) = \alpha_{ij}(r,\theta)d\theta^i \otimes d\theta^j,$$

and $L_{r\partial r}d\eta = 0$ and $L_{r\partial_r}J = 0$ implies that $\alpha_{ij}(\theta)$ does not dependent on r. We have proved (5.14), and obtain the cone formula $g = dr^2 + r^2 g_\Sigma$.

5.4 Special Lagrangian cycles

In this section, we show that the vanishing cycle is special Lagrangian with respect to the Candelas-de la Ossa metric.

5.4.1 Definition of special Lagrangian cycles

We start with a review of calibrated cycles [15]. We say that $\varphi \in \Lambda^k(M)$ on (M,g) is a calibration if for all k-dimensional submanifolds L then

$$\big|\varphi|_L\big|_g \leqslant 1$$

pointwise. The norm is defined as: write $\varphi|_L = f(x)d\mathrm{vol}_L$ and then take $|f(x)|$. Here $d\mathrm{vol}_L$ is the volume form of the induced metric $g|_L$. A submanifold L is calibrated with respect to φ if $\varphi|_L = d\mathrm{vol}_L$.

Let $[L] \in H_k(M,\mathbb{Z})$ be a fixed homology class. Any other representative can be written as $L' = L - \partial\Sigma$. We consider the functional on $[L]$ defined by

$$E(L') = \int_{L'} d\mathrm{vol}_{L'} + \int_\Sigma d\varphi, \quad L' = L - \partial\Sigma. \tag{5.18}$$

In the standard definition of a calibration, we require $d\varphi = 0$ and the bulk term does not appear so that this is the area functional. The more general definition is here for potential future application to non-Kähler geometry.

Proposition 5.7. *Let L be a calibrated submanifold with respect to φ. Then L minimizes the functional $E(L)$ in a given homology class.*

Proof. First, we note

$$E(L) = \int_L \mathrm{dvol}_L = \int_L \varphi$$

When $d\varphi = 0$, this is the topological number $[\varphi] \cdot [L]$. We will show that this is the lower bound of the functional. By Stokes's theorem,

$$E(L') = \int_{L'} \mathrm{dvol}_{L'} + \left[\int_L \varphi - \int_{L'} \varphi \right].$$

Using the calibration property, $\varphi|_{L'} \leqslant |\varphi|_g d\mathrm{vol}|_{L'} \leqslant d\mathrm{vol}|_{L'}$, so

$$E(L') \geqslant \int_{L'} \varphi + \left[\int_L \varphi - \int_{L'} \varphi \right] = \int_L \varphi.$$

□

Let Ω be a holomorphic volume form. Let ω be a hermitian metric, conformally rescaled so that $|\Omega|_\omega = 2^{3/2}$. A special Lagrangian cycle is a submanifold calibrated with respect to $\mathrm{Re}\,\Omega$. The calibration argument implies that special Lagrangian cycles minimize the area functional

$$E(L) = \int_L \mathrm{dvol}_L$$

in a given homology class $[L]$. We now show that $\mathrm{Re}\,\Omega$ is indeed a calibration.

Lemma 5.8. $\mathrm{Re}\,\Omega$ *is a calibration. Furthermore, if L is a calibrated submanifold so that* $\mathrm{Re}\,\Omega|_L = \mathrm{dvol}_L$*, then $\omega|_L = 0$.*

Proof. We closely follow Harvey-Lawson's proof [15]. Let $L \subseteq M$. Suppose v_1, v_2, v_3 are orthonormal vectors spanning T_pL. Harvey-Lawson's identity is

$$|\Omega(v_1, v_2, v_3)|^2 = |v_1 \wedge v_2 \wedge v_3 \wedge Jv_1 \wedge Jv_2 \wedge Jv_3|. \tag{5.19}$$

(The norm of a top form μ is $|\frac{\mu}{\mathrm{dvol}}|$.) Assuming this, then

$$|\Omega(v_1, v_2, v_3)|^2 \leqslant |v_1||Jv_1| \cdots |v_3||Jv_3| = 1.$$

In fact, equality in Hadamard's inequality is achieved if and only if the vectors are orthogonal. This implies $\langle v_i, Jv_k \rangle_g = 0$, which translates to $\omega(v_i, v_k) = 0$ and so $\omega|_L = 0$. The inequality above is

$$|\mathrm{Re}\,\Omega(v_1, v_2, v_3)|^2 + |\mathrm{Im}\,\Omega(v_1, v_2, v_3)|^2 \leqslant 1$$

Therefore, since the v_i are orthonormal, then $|\mathrm{Re}\,\Omega|_L|_g \leqslant 1$ and $\mathrm{Re}\,\Omega$ is a calibration. If equality $\mathrm{Re}\,\Omega|_L = d\mathrm{vol}_L$ holds, then $\omega|_L = 0$ and $\mathrm{Im}\,\Omega|_L = 0$.

We now prove Harvey-Lawson's identity (5.19). We start with the left-hand side. Let $e_1, e_2, e_3, Je_1, Je_2, Je_3$ be an orthonormal basis for T_pM. Since $\Omega|_p$ is an element of the 1 dimensional vector space $\Lambda^{3,0}_p$ it can be written $\Omega|_p = f(p)\varepsilon^1 \wedge \varepsilon^2 \wedge \varepsilon^3$ with

$$\varepsilon_k = \frac{1}{2}(e_k - iJe_k).$$

To find $f(p) \in \mathbb{C}$, we take the norm and use that $|\Omega|_\omega = c$. For suitable normalization ($c = 2^{3/2}$ as will be computed below), we conclude $\Omega|_p = e^{i\theta(p)}\varepsilon^1 \wedge \varepsilon^2 \wedge \varepsilon^3$.

Next, we expand v_i in the basis $\varepsilon_k, \overline{\varepsilon_k}$

$$v_k = A^\ell{}_k \varepsilon_\ell + A^{\bar{\ell}}{}_k \overline{\varepsilon_\ell}.$$

Since $\overline{v_k} = v_k$, using uniqueness of expansion of a basis we obtain that $\overline{A^\ell{}_k} = A^{\bar{\ell}}{}_k$. Since $\Omega|_p = e^{i\theta}\varepsilon^1 \wedge \varepsilon^2 \wedge \varepsilon^3$, we have

$$\Omega(v_1, v_2, v_3) = \Omega(A^\ell{}_1 \varepsilon_\ell, A^m{}_2 \varepsilon_m, A^n{}_3 \varepsilon_n) = e^{i\theta} \det A.$$

Next, we compute the right-hand side of (5.19). Let $e_4 = Je_1$, $e_5 = Je_2$, $e_6 = Je_3$ so that $\{e_i\}$ is an oriented basis. Define a linear map M by its action on this basis

$$M(e_i) := v_i, \quad M(Je_i) := Jv_i.$$

Then

$$|v_1 \wedge Jv_1 \wedge \dots v_3 \wedge Jv_3| = |M(e_1) \wedge M(e_2) \wedge \cdots \wedge M(e_6)| = |\det M|$$

To compute $\det M$, we compute it in the basis $\varepsilon_i, \overline{\varepsilon_i}$. We compute

$$M(\varepsilon_k) = \frac{1}{2}(M(e_k) - iM(Je_k)) = \frac{1}{2}v_k - \frac{i}{2}Jv_k.$$

Using our earlier matrix $A^i{}_k$ and $J\varepsilon_k = i\varepsilon_k$, $J\overline{\varepsilon_k} = -i\overline{\varepsilon_k}$, this is

$$M(\varepsilon_k) = \frac{1}{2}(A^n{}_k \varepsilon_n + A^{\bar{n}}{}_k \overline{\varepsilon_n}) - \frac{i}{2}(iA^n{}_k \varepsilon_n - iA^{\bar{n}}{}_k \overline{\varepsilon_n}) = A^n{}_k \varepsilon_n.$$

Similarly $M(\overline{\varepsilon_k}) = \bar{A}\overline{\varepsilon_k}$, and so in this basis

$$M = \begin{bmatrix} A & 0 \\ 0 & \bar{A} \end{bmatrix}.$$

Therefore $\det M = |\det A|^2$, which proves the identity.

□

A useful formula for calibrated cycles are the special Lagrangian equations. The submanifold L is special Lagrangian if it solves the equations

$$\omega|_L = 0, \quad \operatorname{Im} \Omega|_L = 0. \tag{5.20}$$

These equations imply that L is a calibrated cycle.

Proposition 5.9. *If $L \subseteq (M, \omega, \Omega)$ solves the special Lagrangian equations (5.20), then L is a calibrated cycle. This means we can orient L such that $\operatorname{Re}\Omega|_L = \mathrm{dvol}_L$ and L minimizes the area functional in its homology class.*

Proof. The key identity is that if $\omega|_L = 0$, then

$$\Omega|_L = \frac{e^{i\theta}|\Omega|_\omega}{2^{3/2}} d\mathrm{vol}. \tag{5.21}$$

Therefore the conditions $\omega|_L = 0$, $|\Omega|_\omega = 2^{3/2}$ and $\mathrm{Im}\Omega|_L = 0$ imply $\operatorname{Re}\Omega|_L = \pm d\mathrm{vol}|_L$.

To prove (5.21), fix a point $p \in T_pL$ and an orthonormal basis e_1, e_2, e_3 of T_pL with dual basis e^k. Since $\omega(e_i, e_j) = g(e_i, Je_j)$, the condition $\omega|_L = 0$ implies that

$$\{e_1, e_2, e_3, Je_1, Je_2, Je_3\}$$

is an orthonormal basis of T_pM. Let $\{e^k\}$ the dual basis, and

$$\varepsilon^k = e^k + iJe^k. \tag{5.22}$$

One can check by the definition $\omega(e_i, e_j) = g(e_i, Je_j)$ that $\omega|_p = \sum_k e^k \wedge Je^k$, which in the basis (5.22) is

$$\omega|_p = \frac{i}{2}\sum_k \varepsilon^k \wedge \bar{\varepsilon}^k.$$

The $(3,0)$ form is

$$\Omega|_p = f(p)\varepsilon^1 \wedge \varepsilon^2 \wedge \varepsilon^3$$

for $f(p) \in \mathbb{C}$, since ε^k span $\Lambda^{1,0}_p(M)$ and so any $(3,0)$ form is a multiple of $\varepsilon^1 \wedge \varepsilon^2 \wedge \varepsilon^3$. To identify $f(p)$, we use the formula for the norm

$$i\Omega \wedge \overline{\Omega} = |\Omega|^2_\omega \frac{\omega^3}{3!}.$$

We see that

$$i|f|^2\varepsilon^{123\bar{1}\bar{2}\bar{3}} = \frac{|\Omega|^2_\omega}{2^3} i\varepsilon^{1\bar{1}} i\varepsilon^{2\bar{2}} i\varepsilon^{3\bar{3}}.$$

Therefore $|f|^2 = |\Omega|^2/2^3$ and

$$\Omega|_p = \frac{e^{i\theta(p)}|\Omega|_\omega}{2^{3/2}}\varepsilon^1 \wedge \varepsilon^2 \wedge \varepsilon^3$$

Thus

$$\Omega|_L = \frac{e^{i\theta}|\Omega|_\omega}{2^{3/2}} e^1 \wedge e^2 \wedge e^3,$$

which is (5.21). $\square$

5.4.2 Examples on the smoothing

We give two examples of special Lagrangian cycles on

$$V_t = \{z_1^2 + z_2^2 + z_3^2 + z_4^2 = t\} \subseteq \mathbb{C}^4$$

for $t > 0$, with 3-form

$$\Omega_t = \frac{1}{z_4} dz_1 \wedge dz_2 \wedge dz_3.$$

The Calabi-Yau metric on V_t is the Candelas-de la Ossa metric $\omega_{co,t}$.

• Vanishing cycle $L = S^3$. For this, we use the Candelas-de la Ossa metric

$$\omega_{co,t} = i\partial\bar{\partial} f_t(\tau), \quad \tau = \sum |z_i|^2$$

A special Lagrangian 3-sphere is given by

$$L = \{|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 = t\} \subseteq V_t.$$

If $z_k = x_k + iy_k$, the constraint for V_t implies $t = |x|^2 - |y|^2$, $\langle x, y\rangle = 0$, and the constraint for L is $|x|^2 + |y|^2 = t$. Therefore $y = 0$ and $|x|^2 = t$, so L is topologically a 3-sphere. Since $y = 0$, we see that $(\operatorname{Im}\Omega_t)|_L = 0$. Since τ is constant, we see that $\omega_{co,t}|_L = 0$.

• Special Lagrangian discs. Here we will use the metric $\omega_{Euc} = \sum_k i dz^k \wedge d\bar{z}^k$ restricted to V_t. It is not Calabi-Yau, so this example involves a more general setup. Consider the two discs L_+, L_- given by

$$L_\pm = \{x_1^2 + x_2^2 + x_3^2 \leqslant t\} \subseteq V_t \cap \{y = 0\}.$$

with

$$z_4 = \pm\sqrt{t - x_1^2 - x_2^2 - x_3^2}.$$

These share the same boundary ∂L which is a 2-cycle S^2 which lies on the holomorphic surface $z_4 = 0$. Since $y = 0$, we have $(\operatorname{Im}\Omega_t)|_L = 0$ and $\omega_{Euc}|_L = 0$.

6 Conifold Transitions: Global Geometry

6.1 Overview

Let $\hat{X}$ be a compact Calabi-Yau threefold. A conifold transition $\hat{X} \to X_0 \rightsquigarrow X_t$ is defined as follows:

Step 1: Find holomorphic curves $C_i \subset \hat{X}$ with a neighborhood biholomorphic to the open set $\{\|z\| < 1\}$ in the total space $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$, with $C_i = \{\|z\| = 0\}$. Such curves C_i are called $(-1,-1)$ curves.

Step 2: Contract each C_i to a point p_i and obtain singular space X_0. A change of variables shows that each p_i admits a neighborhood biholomorphic to $\{\sum_{i=1}^4 z_i^2 = 0\} \subseteq \mathbb{C}^4$ (§5.1.2).

Step 3: Realize X_0 as the central fiber of a deformation $\pi : \mathcal{X} \to \Delta$ such that X_t is smooth for $t \neq 0$ and locally near p_i we see the local smoothing $\{\sum z_i^2 = t\}$. For this to exist, we need to satisfy Friedman's condition on the initial curves, and the global smoothing result of Friedman-Tian-Kawamata is stated below. The smoothing has the effect of replacing each $p_i \in X_0$ with 3-sphere S^3 (§5.2).

Theorem 6.1. *[9, 27, 18] Let $\hat{X}$ be a Calabi-Yau threefold with $(-1,-1)$ curves $C_i \subset \hat{X}$. Let $\mu : \hat{X} \to X_0$ be the blow-down map sending C_i to nodal singularities $p_i \in X_0$. If the curves are linearly dependent in homology, so that*

$$\sum_i \lambda_i [C_i] = 0 \in H^4(\hat{X}, \mathbb{C})$$

with $\lambda_i \neq 0$, then X_0 admits a smoothing X_t. This means there is a complex space $\mathcal{X}$ with a proper flat map $\pi : \mathcal{X} \to \Delta$ such that $\pi^{-1}(0) = X_0$ and $\pi^{-1}(t) = X_t$ is a smooth complex manifold for $t \neq 0$.

Let C_i be an initial configuration of $(-1,-1)$ curves on $\hat{X}$ satisfying Friedman's condition $\sum \lambda_i [C_i] = 0$. The theorem above guarantees the existence of a conifold transition $\hat{X} \to X_0 \rightsquigarrow X_t$. Here is a summary of known results on the geometry and topology of such a transition:

• X_t is a compact complex manifold with trivial canonical bundle. [9]

• It is conjectured [1] that all Calabi-Yau threefolds can be connected to each other by conifold transitions.

• However, a conifold transition may produce an X_t which is non-Kähler. (Examples at the end of §6.2) This suggests that these limiting non-Kähler objects should be included in the web of Calabi-Yau threefolds.

• Though non-Kähler in general, X_t is expected to satisfy the ddbar lemma, but as far as I can tell, this is still unknown in full generality [10].

• The topological change is

$$N = k + c,$$

where N is the number of nodes, k is the decrease of b_2, and $2c$ is the increase of b_3. (§6.2)

• [11] There is a sequence of metrics $(\hat{X}, g_a)$ with $d\omega_a^2 = 0$ such that $g_a \to g_0$ uniformly on compact sets for a limiting metric g_0 on X_0 locally modeled near the singularities by a scaling of $g_{co,0}$,

and

$$\mathrm{Vol}(C_i, g_a) \to 0, \quad a \to 0.$$

On the smoothing side, for small t there is a sequence of metrics (X_t, g_t) with $d\omega_t^2 = 0$ and $g_t \to g_0$ uniformly on compact sets with

$$\mathrm{Vol}(L_{i,t}, g_t) \to 0, \quad t \to 0,$$

where $L_{t,i} = \{\|z\|^2 = |t|\} \subseteq V_t \subseteq X_t$ are the vanishing 3-spheres. This result is due to Fu-Li-Yau [11]. Near the singularities, the FLY metrics are modeled by the Candelas-de la Ossa metrics:

$$|g_t - \lambda_i g_{co,t}|_{g_{co,t}} \leqslant C|t|^{2/3}, \quad \text{on } \mathcal{X} \cap \{r < 1\}.$$

Here $r : \mathcal{X} \to (0, \infty)$ is a function which agrees with $r = \|z\|^{2/3}$ near the singular points and $r^{-1}(0) = \mathrm{Sing}(X_0)$. Each ith component of $X_t \cap \{r < 1\}$ is biholomorphic to $V_t \cap \{r < 1\}$, and the metric g_t is close to a λ_i-scaled version of $g_{co,t}$ for $\lambda_i > 0$. For higher order derivatives, the estimate is

$$|\nabla^k_{g_{co,t}}(g_t - \lambda g_{co,t})|_{g_{co,t}} \leqslant C_k |t|^{2/3} r^{-k}, \quad \text{on } \mathcal{X} \cap \{r < 1\}.$$

Let us give a few more details of the Fu-Li-Yau construction. There are two steps:

(A) The construction of a balanced metric ω_0 on X_0 satisfying:

- On $X_0 \backslash \{r < 1\}$, we have $\omega_0 = \omega_{CY}$, where ω_{CY} is a Calabi-Yau metric on $\hat{X}$.

- On $X_0 \cap \{r < \varepsilon\}$, we have $\omega_0 = R\omega_{co,0}$. Here $\varepsilon > 0$ and $R > 1$ are chosen parameters.

- On $X_0 \cap \{\varepsilon \leqslant r \leqslant 1\}$, we have $\omega_0^2 = i\partial\bar{\partial}\beta$ for some $\beta \in \Lambda^{1,1}(\hat{X}, \mathbb{R})$.

Here we go freely between $X_0 \backslash \mathrm{Sing}(X_0)$ and $\hat{X} \backslash \cup C_i$ since these are biholomorphic.

(B) The construction of a balanced metric ω_t on X_t:

This starts by constructing an approximate metric $\hat{\omega}_t$:

- On $X_t \cap \{r < \varepsilon\}$, we have $\omega_t = R\omega_{co,t}$.

- On compact subsets $K \subset \mathcal{X} \backslash \mathrm{Sing}(\mathcal{X})$, we have that $\hat{\omega}_t$ converges to ω_0 smoothly uniformly as $t \to 0$.

This metric $\hat{\omega}_t$ is not balanced, and is globally corrected by

$$\omega_t^2 = \hat{\omega}_t^2 + \theta_t + \bar{\theta}_t.$$

The correction θ comes from solving $E_t(\gamma_t) = \bar{\partial}\hat{\omega}_t^2$ (E_t is the Kodaira-Spencer operator) and $\theta_t = \partial\bar{\partial}^\dagger\partial^\dagger\gamma_t$, $-\bar{\partial}\theta_t = E_t(\gamma_t)$. Fu-Li-Yau show that the correction is θ_t is small: it satisfies $|\theta_t|_{\hat{\omega}_t} \leqslant C|t|^{2/3}$.

• [5] The vanishing cycles $L_{t,i} \subseteq X_t$ can be represented by special Lagrangian 3-spheres with respect to the global geometry g_t. Thus from the point of view of submanifold geometry, the transition exchanges holomorphic 2-cycles with special Lagrangian 3-cycles.

• [4] The compact manifolds X_t for small t admits a pair of metrics (g_t, h_t) solving

$$g^{j\bar{k}} H_{pj\bar{k}} = 0, \quad g^{j\bar{k}} F^p{}_{qj\bar{k}} = 0$$

where H is the 3-form field strength $H = i(\partial - \bar{\partial})\omega$ and F is the 2-form field strength $F = \bar{\partial}(h^{-1}\partial h)$. In particular, X_t admits balanced metrics and has stable tangent bundle. (The equation $g^{j\bar{k}} H_{pj\bar{k}} = 0$ is equivalent to $d\omega^2 = 0$.) Note that Calabi-Yau metrics g_{CY} solve these equations for $h = g = g_{CY}$, so these are generalizations of Calabi-Yau metrics on the non-Kähler spaces reached by conifold transitions.

• It is conjectured by S.-T. Yau that the pair (g, h) can be further deformed to solve the Strominger system:

$$d(|\Omega|_\omega \omega^2) = 0, \quad g^{j\bar{k}} F^p{}_{qj\bar{k}} = 0$$
$$i\partial\bar{\partial}\omega = \alpha'(\operatorname{Tr} R_\omega \wedge R_\omega - \operatorname{Tr} F_h \wedge F_h). \tag{6.1}$$

The equation $d(|\Omega|_\omega \omega^2) = 0$ can be solved by conformally changing the Fu-Li-Yau metric on X_t, but whether (6.1) is solvable through conifold transitions is unknown.

6.2 Topological change

In this section, we follow the exposition given in Rossi's survey [23].

Let $\hat{X} \to \underline{X} \rightsquigarrow X$ be a conifold transition with N nodes. We denote the N holomorphic curves which are being contracted by $C_i \subset \hat{X}$, and $[C_i] \in H_2(\hat{X}, \mathbb{R})$ their homology class. A transition decreases b_2 and increases b_3, and we define the jumps in homology rank by

$$b_2(X_t) = b_2(\hat{X}) - k, \quad b_3(X_t) = b_3(\hat{X}) + 2c.$$

With this notation, the fundamental identity for topology change to be explained in this section is:

$$N = k + c. \tag{6.2}$$

and k is also equal to

$$k = \dim \text{ subspace in } H_2(\hat{X}, \mathbb{R}) \text{ generated by } [C_i]. \tag{6.3}$$

As a consequence of $N = k + c$, we see that if a transition exists then there must be a linear dependence relation between the N curves $[C_i]$. Viewed another way, we can deduce the change in b_3 by the initial position of the 2-cycles C_i by $c = N - k$. Also, $c \geqslant 1$ is the number of independent vanishing 3-cycles in X_t.

We will need the following properties:

• There is a neighborhood $\hat{U}_i$ of each $(-1,-1)$-curve C_i in $\hat{X}$ which is diffeomorphic to $S^2 \times B^4$. This means that though the bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$ is not holomorphically trivial, it is in fact a trivial rank 4 real vector bundle. For the diffeomorphism to $S^2 \times B^4$, see [23].

• There is a neighborhood $U_{t,i}$ of each vanishing cycle L_i in X_t which is diffeomorphic to $S^3 \times B^3$. We discussed earlier how the equations of V_t can be written as $t = |x|^2 - |y|^2$, $\langle x, y \rangle = 0$. Let $p = x/(t + \|y\|^2)^{1/2}$, so that $(p, x) \in \mathbb{R}^4 \times \mathbb{R}^4$ satisfy $\|p\| = 1$, $\langle p, y \rangle = 0$. This is diffeomorphic to the total space of TS^3, which is known to be a trivial bundle over S^3. (A construction of a global basis of sections of TS^3 can be constructed using quaternions.)

• Let $\hat{U}_i \subseteq \hat{\underline{U}}_i$ be two nested neighborhoods of a $(-1,-1)$-curve in $\hat{X}$ as above. Then the annulus $\hat{A} = \hat{U}_i \cap \hat{\underline{U}}_i$ is homotopic to $S^2 \times S^3$. This is because $\hat{V} \backslash \hat{U}_i$ is diffeomorphic to the cone $V_0 \backslash \{\|z\| < 1\}$, whose cross-section is $S^3 \times S^2$. Similarly, $A_t = U_{i,t} \cap \underline{U}_{i,t} \subseteq X_t$ is homotopic to $S^2 \times S^3$.

Let $\hat{U} = \cup_{i=1}^{N} \hat{U}_i \subseteq \hat{X}$ be an open set containing all curves C_i with $\hat{U} = \sqcup S^2 \times B^4$, and $U_t = \cup_{i=1}^{N} U_{t,i} \subseteq X_t$ be an open set containing all vanishing cycles with $U_t = \sqcup S^3 \times B^3$. We now compute the topology of X_t based on the knowledge of $\hat{X}$. We start with:

Lemma 6.2. $b_1(\hat{X}) = b_1(X_t)$.

Proof. Recall the Meyer-Vietoris sequence: if $X = U \cup V$ then

$$\cdots \to H_i(U \cap V) \to H_i(U) \oplus H_i(V) \to H_i(X) \to H_{i-1}(U \cap V) \to H_{i-1}(U) \oplus H_{i-1}(V) \ldots$$

We apply this to $\hat{X} = (\hat{X} \backslash \hat{U}) \cup \underline{\hat{U}}$, so that

$$H_1(\sqcup S^3 \times S^2) \to H_1(\hat{X} \backslash \hat{U}) \oplus H_1(\sqcup S^2) \to H_1(\hat{X}) \to H_0(\sqcup S^3 \times S^2) \to H_0(\hat{X} \backslash \hat{U}) \oplus H_0(\sqcup S^2)$$

while on $\underline{X}$, we have

$$H_1(\sqcup S^3 \times S^2) \to H_1(\underline{X} \backslash U) \oplus H_1(\sqcup \{pt\}) \to H_1(\underline{X}) \to H_0(\sqcup S^3 \times S^2) \to H_0(\underline{X} \backslash U) \oplus H_0(\sqcup \{pt\})$$

and on X_t we have

$$H_1(\sqcup S^3 \times S^2) \to H_1(X_t \backslash U_t) \oplus H_1(\sqcup S^3) \to H_1(X_t) \to H_0(\sqcup S^3 \times S^2) \to H_0(X_t \backslash U_t) \oplus H_0(\sqcup S^3)$$

To connect these diagrams, we use the contraction maps $\hat{X} \to \underline{X}$ and $X_t \to \underline{X}$, and that $\hat{X} \backslash \hat{U}$ and $X_t \backslash U_t$ are both diffeomorphic to $\underline{X} \backslash U$ (as they are a smooth family of complex structures). The 5-lemma implies that $H_1(\hat{X}) = H_1(\underline{X}) = H_1(X_t)$. Recall the 5-lemma: given a diagram between two exact sequence

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & D \arrow[r] \arrow[d, "i"] & E \arrow[d, "j"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

If g, i are isomorphisms, f surjective and j injective, then h is an isomorphism. $\square$

Since we are assuming that $\hat{X}$ is a simply connected Calabi-Yau threefold, it follows that $b_1(X_t) = 0$ on the other side of a conifold transition.

Lemma 6.3. *$b_2(\hat{X}) = b_2(X_t) - \kappa$, where κ is the dimension of the subspace in $H_2(\hat{X}, \mathbb{R})$ spanned by the $[C_i]$.*

Proof. Let $C = \cup C_i \subset \hat{X}$ be degenerating 2-cycles and $L = \cup L_i \subset X_t$ be the vanishing 3-cycles. The long exact sequence for relative homology gives

$$\cdots \to H_2(C) \xrightarrow{\iota *} H_2(\hat{X}) \to H_2(\hat{X}, C) \to H_1(C) \xrightarrow{\iota *} \cdots \tag{6.4}$$

Recall that relative homology $H(\hat{X}, C)$ means: homology of space where we identify all points in C to be a single point (or cycles $\alpha \in Z_n(\hat{X})$ such that $\partial \alpha \in Z_{n-1}(C)$). But we will not need to have intuition for this definition here; the only properties we will use is the exact sequence above, and the Lefschetz duality

$$H_i(\hat{X}, C) \cong H_c^{n-i}(\hat{X} \backslash C)$$

which holds for topological manifolds and implies in our setup that

$$H_i(\hat{X}, C) \cong H_i(X_t, L), \tag{6.5}$$

since $\hat{X}\backslash C \cong X_t\backslash L$. The long exact sequence for relative homology gives

$$\bigsqcup_{i=1}^{N}[C_i] \xrightarrow{\iota_*} H_2(\hat{X}) \xrightarrow{\varphi} H_2(\hat{X}, C) \to 0.$$

By the rank-nullity theorem and (6.5),

$$\begin{aligned} \dim H_2(\hat{X}) &= \dim H_2(\hat{X}, C) + \dim\ker\varphi \\ &= \dim H_2(X_t, L) + \dim\operatorname{im}\iota_*. \end{aligned} \tag{6.6}$$

If κ is the dimension of the subspace in $H_2(\hat{X})$ generated by $[C_i]$, we conclude

$$\kappa = b_2(\hat{X}) - \dim H_2(X_t, L). \tag{6.7}$$

Since $H_2(X_t, L) \cong H_2(X_t)$, we obtain

$$\kappa = b_2(\hat{X}) - b_2(X_t). \tag{6.8}$$

This can be seen by the exact sequence

$$H_2(L) \to H_2(X_t) \to H_2(X_t, L) \to H_1(L)$$

which for $L = \sqcup_{i=1}^{N} S^3$ implies $H_2(X_t) \cong H_2(X_t, L)$. □

Lemma 6.4. *$N = k + c$*

Proof. Recall the Euler characteristic $\chi = \sum_{k=1}^{6}(-1)^k b_k$. The excision property for the Euler characteristic is: for $U \subseteq X$ open,

$$\chi(X\backslash U) = \chi(X) - \chi(U).$$

In our setup, this gives

$$\chi(\hat{X}\backslash\hat{U}) = \chi(\hat{X}) - N\chi(S^2 \times B^4)$$

and

$$\chi(X_t\backslash U_t) = \chi(X_t) - N\chi(S^3 \times B^3).$$

Since $\hat{X}\backslash\hat{U}$ and $X_t\backslash U_t$ are diffeomorphic, and $\chi(S^2) = 2$ and $\chi(S^3) = 0$, plus $\chi(S^2 \times B^4) = \chi(S^2)$ by deformation retraction, it follows that

$$\chi(\hat{X}) - 2N = \chi(X_t).$$

We proved that $\hat{X}$ and X_t have $b_1 = 0$, so $\chi = 2 + 2b_2 - b_3$ and we obtain

$$N = (b_2(\hat{X}) - b_2(X_t)) + \frac{1}{2}(b_3(X_t)) - b_3(\hat{X})).$$

□

Remark 6.5. If the initial manifold has $b_2(\hat{X}) = 1$ (if for example $\hat{X}$ is a quintic threefold in $\mathbb{P}^4$), then two homologically linearly dependent curves C_1, C_2 produce a conifold transition where the resulting manifold has $b_2(X_t) = 0$. A compact Kähler manifold cannot have $b_2 = 0$, since ω represents a non-zero cohomology class in H^2. We see that conifold transitions possibly take us out of Kähler geometry.

Example 6.6. Here is another example of a non-Kähler transition from [2]. Start with a small resolution $\hat{X}$ of a singular quintic $P = Z_3\, G(Z_0, \ldots, Z_4) + Z_4\, H(Z_0, \ldots, Z_4) = 0$. This has $h^{1,1} = 2$, and there are 16 $(-1,-1)$ curves $C_i \subseteq \hat{X}$ from the small resolution, but they are all linearly dependent. Suppose they are generated by C_1. Take 2 curves C_1, C_2; they satisfy Friedman's condition so we can produce a transition $\hat{X} \to \underline{X} \rightsquigarrow X_t$. Now the other 14 curves are trivial in homology $[C_i] = [\partial D_i]$. Furthermore by [Remark 3.2.8, Lemma 3.3.1] in McDuff-Salamon [22], the $(-1,-1)$ curves deform along the family so that $C_{i,t}$ are holomorphic curves. If X_t admits a Kähler metric ω_t, then

$$0 < \int_{C_{i,t}} \omega_t = \int_{D_{i,t}} d\omega_t = 0,$$

which is a contradiction. So X_t is non-Kähler, but $b_1(X_t) = 1$.

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