Yang-Mills connections

- $(E, H) \to (M, g)$ complex vector bundle with metric H

$\mathcal{H} = \{$ metric compatible connections on $(E, H)\}$

$$I : \mathcal{H} \to \mathbb{R}, \quad I(A) = \int_M |F(A)|^2 \, d\mathrm{vol}_g$$

We will look for connections that are critical pts of this functional.

- Notation:

$$d\mathrm{Vol} = \sqrt{\det g_{ij}} \; dx^1 \wedge \cdots \wedge dx^n$$

$$|F|^2 = \sum_{a,b} g^{ij} g^{k\ell} F_{ik}{}^{a}{}_{b} \overline{F_{j\ell}{}^{a}{}_{b}}$$

$F_{ik}{}^{a}{}_{b}$: representative in orthormal frame $\{e_a\}$ of (E, H)

$$F = \tfrac{1}{2} F_{ik}{}^{a}{}_{b} \, dx^i \wedge dx^k \otimes e_a \otimes e^b.$$

$F_{ik}{}^{a}{}_{b} = - \overline{F_{ik}{}^{b}{}_{a}}$ (showed earlier for metric comp ∇)

Can write: $|F|^2 = -\mathrm{Tr}\, F_{ik} F^{ik}$, since $F_{ik}^* = -F_{ik}$

(Norm of matrix is $|C|^2 = \mathrm{Tr}\, C^* C = -\mathrm{Tr}\, C^2$ if $C^* = -C$)

- $A_0 \in \mathcal{H}$ is a critical pt of I if $\forall$ paths $A(t) \in \mathcal{H}$ with $A(0) = A_0$, then

$$\left.\frac{d}{dt}\right|_{t=0} I(A(t)) = 0.$$

Critical pt eqn:

$$\frac{d}{dt} I(A(t)) = -2 \int_M \mathrm{Tr} \frac{d}{dt} F_{ik} \, F^{ik} \, d\mathrm{vol}_g.$$

Showed earlier: $\frac{d}{dt} F = d_\nabla a, \quad a = \frac{d}{dt} A$

In components:

$$d_\nabla a = \nabla_i a_k{}^\alpha{}_\beta \, dx^i \wedge dx^k \otimes e_\alpha \otimes e^\beta$$

$$d_\nabla a = \frac{1}{2}\left(\nabla_i a_k - \nabla_k a_i\right) dx^i \wedge dx^k, \quad a \in \Omega^1(\mathrm{End}\,E)$$

anti-sym (ik)

connection here acts on all indices

$$\nabla_i a_k{}^\alpha{}_\beta = \partial_i a_k{}^\alpha{}_\beta - \underbrace{\Gamma_{ik}^p}_{\text{sym}} a_p{}^\alpha{}_\beta + A_i{}^\alpha{}_\mu a_k{}^\mu{}_\beta - a_k{}^\alpha{}_\mu A_i{}^\mu{}_\beta$$

This consistent with old formula

$$d_\nabla a = da + A \wedge a + a \wedge A. \quad (\Gamma_{ik}^p - \Gamma_{ki}^p \text{ cancels})$$

$$\Rightarrow \frac{d}{dt} I(A(t)) = -2\int_M \mathrm{Tr}\,(\nabla_i a_k - \nabla_k a_i) F^{ik} \, dvol_g$$

$$= -4\int_M \mathrm{Tr}\, \nabla_i a_k F^{ik} \, dvol_g$$

integrate by parts.

Divergence Thm: $\int_M \nabla_i V^i \, dvol_g = 0, \quad \forall V \in \Gamma(TM)$, ∇ = Levi-Civita.

Apply to: $\int_M \nabla_i \big(\underbrace{\mathrm{Tr}\, a_k F^{ik}}_{\in \Gamma(TM)}\big) \, dvol_g = 0.$

$$\Rightarrow \frac{d}{dt} I(A(t)) = 4\int_M \mathrm{Tr}\, a_k \nabla_i F^{ik} \, dvol_g.$$

Note: $\nabla_k \mathrm{Tr}\, B = \mathrm{Tr}\, \nabla_k B$, $B \in \Gamma(\mathrm{End}\,E)$:

$$\mathrm{Tr}\,\nabla_k B = \mathrm{Tr}\,\partial_k B + \cancel{\mathrm{Tr}\, AB} = \partial_k \mathrm{Tr} B - \cancel{\mathrm{Tr}\, BA}$$

If $\frac{d}{dt}\Big|_{t=0} I(A(t)) = 0$ $\forall$ paths $A(t)$ with $A(0) = A$,

then $\nabla_i F^{ik} = 0$. More details: consider the path $A(t) = A + ta$, $a_j = \nabla^k F_{kj}$. Check $A(t) \in \mathcal{A}$:

$A + a$ admissible if $a \in \Omega^1(\mathrm{End}\,E)$, $a^* = -a$

$$0 = \int_M g^{ij} \,\mathrm{Tr}\, \nabla^k F_{ki} \nabla^k F_{kj} \, dvol_g$$

$$\Rightarrow \quad 0 = \nabla^K F_{Ki}{}^{\alpha}{}_{\beta} \qquad \text{Yang-Mills eqn}$$

Divergence Thm vs Stokes's Thm:

Will show $\int_M \nabla_i V^i \, d\mathrm{vol}_g = 0.$

claim: $\nabla_i V^i = \dfrac{1}{\sqrt{\det g}} \partial_i \left(\sqrt{\det g}\, V^i \right).$

Assuming claim:

$$(\nabla_i V^i)\sqrt{\det g}\, dx^1 \wedge \dots \wedge dx^n = \partial_i \left(\sqrt{\det g}\, V^i\right) dx^1 \wedge \dots \wedge dx^n$$

$$= d\left(\sum_i (-1)^{i+1} \sqrt{\det g}\, V^i \, dx^1 \wedge \dots \widehat{dx^i} \wedge \dots \wedge dx^n \right)$$

$$= d\left(\iota_V \, d\mathrm{vol}_g \right)$$

$$\Rightarrow \int_M \nabla_i V^i \, d\mathrm{vol}_g = \int_M d\left(\iota_V \, d\mathrm{vol}_g \right) = 0 \qquad \text{Stokes}$$

Proof of claim:

$$\frac{1}{\sqrt{\det g}} \partial_i \left(\sqrt{\det g}\, V^i \right) = \frac{1}{\sqrt{\cancel{\det g}}} \frac{1}{2\sqrt{\cancel{\det g}}} \cancel{\det g} \left(g^{pq} \partial_i g_{pq} \right) V^i$$

$$+ \partial_i V^i$$

$$\nabla_i V^i = \partial_i V^i + \Gamma^i_{ip} V^p$$

$$= \partial_i V^i + \frac{1}{2} g^{il} \left(-\partial_l \cancel{g_{ip}} + \partial_i \cancel{g_{lp}} + \partial_p g_{il} \right) V^p$$

$$= \partial_i V^i + \frac{1}{2} \left(g^{pq} \partial_i g_{qp} \right) V^i \quad \checkmark$$

Hodge Star:

Let (M, g) Riemannian mfd, dim n. Define:

$*: \Omega^k(M) \to \Omega^{n-k}(M)$ linear map s.t.

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle_g \, dvol_g \quad \forall \alpha, \beta \in \Omega^k(M).$$

Using this notation, Yang-Mills functional is:

$$I(A) = -\int_M \mathrm{Tr}\, F \wedge *F$$

$$-\mathrm{Tr}\, F_{ij} F^{ij} \, dvol_g = -\mathrm{Tr}\, F \wedge *F$$

ex) $(\mathbb{R}^4, g_{Euc})$, coords (x^0, x^1, x^2, x^3).

$*(dx^0 \wedge dx^1) = ?$

$*(dx^0 \wedge dx^1) = \sum_{i<j} a_{ij}\, dx^i \wedge dx^j$.

$$dx^0 \wedge dx^1 \wedge *(dx^0 \wedge dx^1) = \langle dx^{01}, dx^{01} \rangle \, dvol_g$$
$$= dx^{0123}$$

$\Rightarrow a_{23} = 1$

$dx^0 \wedge dx^2 \wedge *(dx^0 \wedge dx^1) = \langle dx^{02}, dx^{01} \rangle \, dvol_g = 0$.
$\Rightarrow a_{13} = 0$.

Continuing in this way $\rightsquigarrow$ $*dx^{01} = dx^{23}$.

Computing more examples gives

$*dx^{01} = dx^{23}$, $\quad *dx^{02} = -dx^{13}$, $\quad *dx^{03} = dx^{12}$
$*dx^{12} = dx^{03}$, $\quad *dx^{13} = -dx^{02}$, $\quad *dx^{23} = dx^{01}$

Using linear combinations of this, we can compute $*\alpha$ for any $\alpha \in \Omega^2(\mathbb{R}^4, g_{Euc})$. We can then direct check the following identities:

- $**\alpha = \alpha \quad \forall \alpha \in \Omega^2(\mathbb{R}^4, g_{Euc})$

(In general: $**\alpha = (-1)^{k(n-k)}\alpha$, $\dim M = n$, $k = \deg \alpha$)

- $*\alpha \wedge *\alpha = \alpha \wedge \alpha \quad \forall \alpha \in \Omega^2(\mathbb{R}^4, g_{Euc})$

$\left(*\alpha \wedge *\alpha = \langle \alpha, *\alpha \rangle \, dvol_g = \alpha \wedge **\alpha = \alpha \wedge \alpha \right)$

Since $*^2 = 1$, 2-forms split into ± 1 eigenspaces.

$*: \Omega^2(\mathbb{R}^4) \to \Omega^2(\mathbb{R}^4)$, with eigenvectors:

$\omega_i^+ = dx^0 \wedge dx^i + dx^j \wedge dx^k$, $\quad \varepsilon_{ijk} = 1$

$\omega_i^- = dx^0 \wedge dx^i - dx^j \wedge dx^k$

check: $*\omega_i^+ = \omega_i^+$ self-dual

$*\omega_i^- = -\omega_i^-$ anti-self-dual

Yang-Mills in 4D:

$$Q = \frac{1}{8\pi^2} \int_{M^4} \mathrm{Tr}\, F \wedge F$$

Topological charge indep of connection (by property of $ch_2(E)$)

claim: $8\pi^2 |Q| \leq I(A)$.

Indeed:

$$0 \leq \int_{M^4} |F \pm *F|^2 \, dvol_g$$

$$= -\int_{M^4} \mathrm{Tr}\, (F \pm *F) \wedge * (F \pm *F)$$

$$= -\int_{M^4} \mathrm{Tr}\, F \wedge *F \mp \int_{M^4} \mathrm{Tr}\, F \wedge **F$$

$$\mp \int_{M^4} \mathrm{Tr}\, *F \wedge *F - \int_{M^4} \mathrm{Tr}\, *F \wedge **F$$

Use: $**F = F$, $*F \wedge *F = F \wedge F$.

$$0 \leq -2\int \mathrm{Tr}\, F \wedge *F \mp 2\int \mathrm{Tr}\, F \wedge F$$

$$\underbrace{\pm \int_{M^4} \mathrm{Tr}\, F \wedge F}_{\pm 8\pi^2 Q} \leq \underbrace{-\int_{M^4} \mathrm{Tr}\, F \wedge *F}_{I(A)}$$

$$8\pi^2 |Q| \leq I(A).$$

Note: if $*F = F$ or $*F = -F$, then the above calculation shows

$$8\pi^2 |Q| = I(A).$$

$\Rightarrow$ If $*F = F$ or $*F = -F$, $\nabla = d + A$ attains minimum of functional $I(A)$.

$\Rightarrow$ Connections solving $*F = F$ or $*F = -F$ are critical pts of $I(A)$, i.e. they solve the Yang-Mills eqn.

Say $\nabla = d + A$ is self-dual if: $*F = F$
anti-self-dual if: $*F = -F$.

$$\left(\begin{array}{c} \text{2nd order eqn} \\ \nabla^i F_{ik} = 0 \end{array} \right) \Leftarrow \left(\begin{array}{c} \text{1st order eqn} \\ *F = -F \\ \text{or } *F = F \end{array} \right)$$