Review of the exterior derivative:

$$d : \Omega^k(M) \to \Omega^{k+1}(M)$$

$$\alpha \overset{\text{loc}}{=} \frac{1}{k!} \alpha_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

$$d\alpha \overset{\text{loc}}{=} \frac{1}{k!} \partial_p \alpha_{i_1 \cdots i_k} dx^p \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

ex) $\alpha = \alpha_i dx^i \in \Omega^1(M)$

$$d\alpha = \partial_k \alpha_i \, dx^k \wedge dx^i.$$

But components are $d\alpha = \frac{(d\alpha)_{ik}}{2} dx^i \wedge dx^k$

with $(d\alpha)_{ik} = -(d\alpha)_{ki}$.

$$d\alpha = \frac{1}{2}(\partial_k \alpha_i - \partial_i \alpha_k) dx^k \wedge dx^i$$

$$\Rightarrow (d\alpha)_{ki} = \partial_k \alpha_i - \partial_i \alpha_k.$$

Need to check: $(d\alpha)_{ki}$ is a tensor, i.e.

$$(d\alpha)_{ki} = \frac{\partial \tilde{x}^\mu}{\partial x^k} \frac{\partial \tilde{x}^\nu}{\partial x^i} (d\tilde{\alpha})_{\mu\nu}, \quad \tilde{\alpha}_\mu = \frac{\partial x^i}{\partial \tilde{x}^\mu} \alpha_i.$$

Indeed:

$$\frac{\partial \alpha_i}{\partial x^k} - \frac{\partial \alpha_k}{\partial x^i} = \frac{\partial}{\partial x^k}\left(\frac{\partial \tilde{x}^\mu}{\partial x^i} \tilde{\alpha}_\mu\right) - \frac{\partial}{\partial x^i}\left(\frac{\partial \tilde{x}^\mu}{\partial x^k} \tilde{\alpha}_\mu\right)$$

cancellation of mixed partials

$$= \frac{\partial \tilde{x}^\mu}{\partial x^i} \frac{\partial}{\partial x^k} \tilde{\alpha}_\mu - \frac{\partial \tilde{x}^\mu}{\partial x^k} \frac{\partial}{\partial x^i} \tilde{\alpha}_\mu$$

$$= \frac{\partial \tilde{x}^\mu}{\partial x^i} \frac{\partial \tilde{x}^\nu}{\partial x^k} \frac{\partial}{\partial \tilde{x}^\nu} \tilde{\alpha}_\mu - \frac{\partial \tilde{x}^\mu}{\partial x^k} \frac{\partial \tilde{x}^\nu}{\partial x^i} \frac{\partial}{\partial \tilde{x}^\nu} \tilde{\alpha}_\mu$$

$$= \frac{\partial \tilde{x}^\mu}{\partial x^k} \frac{\partial \tilde{x}^\nu}{\partial x^i} (d\tilde{\alpha})_{\mu\nu}.$$

Same calculation: $d f^* \alpha = f^* d\alpha$

$$(f^*\alpha)_i(x) = \frac{\partial f^\mu}{\partial x^i}\,\alpha_\mu(f(x)),$$

$$(f^* d\alpha)_{ki}(x) = \frac{\partial f^\mu}{\partial x^k}\frac{\partial f^\nu}{\partial x^i}\left(\partial_\mu \alpha_\nu - \partial_\nu \alpha_\mu\right)(f(x))$$

$$(d f^*\alpha)_{ij}(x) = \frac{\partial}{\partial x^i}\left(\frac{\partial f^\mu}{\partial x^j}\alpha_\mu(f(x))\right) - (i \leftrightarrow j)$$

cancels

$$= \frac{\partial f^\mu}{\partial x^j}\frac{\partial f^\nu}{\partial x^i}\frac{\partial}{\partial x^\nu}\alpha_\mu(f(x)) - (i \leftrightarrow j)$$

$$= (f^* d\alpha)_{ij}(x).$$

Formula without indices:

- $\alpha \in \Omega^1(M)$
- $d\alpha(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y])$.

check:

$$(d\alpha)_{ij}X^iY^j = X^i\partial_i(Y^j\alpha_j) - Y^i\partial_i(X^j\alpha_j) - \alpha_j(X^i\partial_i Y^j - Y^i\partial_i X^j)$$

$$= (\partial_i\alpha_j - \partial_j\alpha_i)X^iY^j$$

consistent with coord defn ✓

Prop: $d^2 = 0$.

check: $$d^2\alpha = \underbrace{\partial_q\partial_p}_{\text{symmetric}}\frac{1}{k!}\alpha_{i_1\cdots i_k}\underbrace{dx^q\wedge dx^p}_{\text{anti-symmetric}}\wedge dx^{i_1}\cdots\wedge dx^{i_k}$$

$$= 0.$$

Prop: $d(\alpha\wedge\beta) = d\alpha\wedge\beta + (-1)^k\alpha\wedge d\beta$

where $\alpha\in\Omega^k(M)$.

check: $\alpha = \frac{1}{k!}\alpha_K dx^K$, $\beta = \frac{1}{J!}\beta_J dx^J$ multi-index notation

$$d\left(\frac{1}{K!}\alpha_K dx^K \wedge \frac{1}{J!}\beta_J dx^J\right)$$

$$= \frac{1}{K!J!}\partial_p \alpha_K \beta_J \, dx^p \wedge dx^K \wedge dx^J$$

$$+ \frac{1}{K!}\frac{1}{J!}\alpha_K \partial_p \beta_J \, dx^p \wedge dx^K \wedge dx^J$$

need to switch: pick up $(-1)^K$

Curvature

Let $\nabla = d + A$ be connection on $E \to M$.

curvature (or field strength) of ∇ is:

$$F = dA + A \wedge A.$$

Expanded in components:

$$A = A_i{}^\alpha{}_\beta \, dx^i \otimes e_\alpha \otimes e^\beta$$

$$F := \frac{1}{2} F_{ij}{}^\alpha{}_\beta \, dx^i \wedge dx^j \otimes e_\alpha \otimes e^\beta$$

$$F^\alpha{}_\beta = \partial_i A_j{}^\alpha{}_\beta \, dx^i \wedge dx^j + (A_i{}^\alpha{}_\gamma dx^i) \wedge (A_j{}^\gamma{}_\beta dx^j)$$

omit form indices

$$= \frac{1}{2}(\partial_i A_j{}^\alpha{}_\beta - \partial_j A_i{}^\alpha{}_\beta) \, dx^i \wedge dx^j$$

anti-symmetrize

$$+ \frac{1}{2}(A_i{}^\alpha{}_\gamma A_j{}^\gamma{}_\beta - A_j{}^\alpha{}_\gamma A_i{}^\gamma{}_\beta) \, dx^i \wedge dx^j$$

$$F_{ij}{}^\alpha{}_\beta = \partial_i A_j{}^\alpha{}_\beta - \partial_j A_i{}^\alpha{}_\beta + A_i{}^\alpha{}_\gamma A_j{}^\gamma{}_\beta - A_j{}^\alpha{}_\gamma A_i{}^\gamma{}_\beta$$

$$F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]$$

omitting End indices

Transformation Law: triv U, V, trans $c_{UV} : U \cap V \to GL(k)$,

- $s_U = c_{UV}\, s_V$ $\qquad s \in \Gamma(E)$
- $(A_U)_i = c_{UV} (A_V)_i c_{UV}^{-1} - \partial_i c_{UV}\, c_{UV}^{-1}$
- $(F_U)_{ij} = c_{UV} (F_V)_{ij}\, c_{UV}^{-1}$

Consequently: $F \in \Omega^2 \otimes \Gamma(\mathrm{End}\, E)$.

$\in \Gamma(\mathrm{End}\, E)$

$\uparrow$ recall $S = S^\alpha{}_\beta$

$(S_U)^\alpha{}_\beta = c_{UV}{}^\alpha{}_\nu (S_V)^\nu{}_\mu\, c_{VU}{}^\mu{}_\beta$

$S_U = c_{UV}\, S_V\, c_{VU}$

If $F=0$ in one frame then $F=0$ in any other frame

Simplest case: line bundle $L \to M$

: c_{UV}, A_i, F_{ij} all 1×1 matrices

: $F = dA$

transformation law for line bundles
$$\begin{cases} A \mapsto A + d\phi, & c_{UV} = e^{-\phi} \\ F \mapsto F & \Rightarrow F \in \Omega^2(M). \end{cases}$$

Proof of transformation law:

$$F_U = dA_U + A_U \wedge A_U$$

$$= d\left(c_{UV} A_V c_{UV}^{-1} - dc_{UV}\, c_{UV}^{-1}\right)$$

$$+ \left(c_{UV} A_V c_{UV}^{-1} - dc_{UV}\, c_{UV}^{-1}\right) \wedge \left(c_{UV} A_V c_{UV}^{-1} - dc_{UV}\, c_{UV}^{-1}\right)$$

d cross 1-form A

$$= dc_{UV} \cancel{A_V} c_{UV}^{-1} + c_{UV}\, dA_V\, c_{UV}^{-1} - c_{UV} A_V\, dc_{UV}^{-1}$$

$$- \cancel{d^2 c_{UV}}\, c_{UV}^{-1} + dc_{UV} \wedge dc_{UV}^{-1}$$

$$+ c_{UV} A_V \wedge A_V c_{UV}^{-1} - c_{UV} A_V c_{UV}^{-1} dc_{UV}\, c_{UV}^{-1}$$

$$- dc_{UV} \wedge \cancel{A_V} c_{UV}^{-1} + (dc_{UV}\, c_{UV}^{-1})^2$$

$$dc_{UV}^{-1} = - c_{UV}^{-1} dc_{UV}\, c_{UV}^{-1}$$

$$= c_{uv}\, dA_v\, c_{uv}^{-1} + \cancel{c_{uv} A_v c_{uv}^{-1} dc_{uv} c_{uv}^{-1}}$$
$$\cancel{- dc_{uv} c_{uv}^{-1} dc_{uv} c_{uv}^{-1}} + c_{uv} A_v \wedge A_v c_{uv}^{-1}$$
$$\cancel{- c_{uv} A_v c_{uv}^{-1} dc_{uv} c_{uv}^{-1}} + \cancel{(dc_{uv} c_{uv}^{-1})^2}$$

$$F_u = c_{uv} \, (\underbrace{dA_v + A_v \wedge A_v}_{F_v}) \, c_{uv}^{-1}$$

<u>Prop</u>: Curvature is obstruction to finding local triv with constant transition fun c_{uv}.

$E \to M$ admits $\nabla = d + A$ with $F_A \equiv 0$.	$\iff$	E admits local triv $M = \bigcup \mathcal{U}_\alpha$ with $c_{\mathcal{U}_\alpha \mathcal{U}_\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to Gl(k)$ constant matrices.

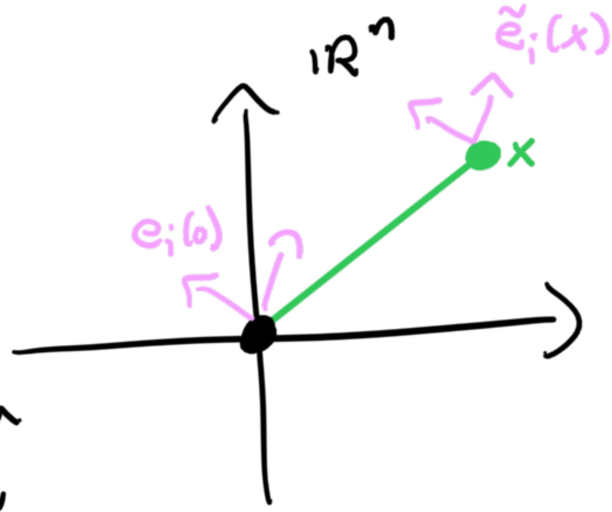
ref: Donaldson-Kronheimer

<u>Pf</u>: ($\Leftarrow$) easy: in such a triv, $\nabla = d$ is well-defn. $A = 0$ transforms correctly:
$$A_u = c_{uv} A_v c_{uv}^{-1} - d\cancel{c_{uv}} c_{uv}^{-1}$$

($\Rightarrow$) Suppose $\nabla = d + A$ has $F \equiv 0$.
Let $\mathcal{U}$ be triv with local frame $\{e_i\}$.
Will construct new frame $\{\tilde{e}_i\}$ with $\tilde{A} \equiv 0$.
Then $E \to M$ admits triv where $\nabla = d$.
By $\tilde{A}_u = \tilde{c}_{uv} \tilde{A}_v \tilde{c}_{uv}^{-1} - d\tilde{c}_{uv} \tilde{c}_{uv}^{-1}$
$\Rightarrow d\tilde{c}_{uv} \equiv 0 \Rightarrow \tilde{c}_{uv} \equiv$ const matrix.

New frame: From origin $o \in \mathcal{U}$, parallel transport $\{e_i(o)\}$ along radial lines.

New frame with $D_t \tilde{e}_a = 0$.



Fix x, let $\gamma(t) = tx$.

$$D_t = \dot{\gamma}^i \nabla_i = x^i \nabla_i$$

$$r\frac{\partial}{\partial r} = x^i \partial_i, \quad \nabla_{r\frac{\partial}{\partial r}} \tilde{e}_a = x^i \nabla_i \tilde{e}_a$$

$$D_t \tilde{e}_a = 0 \Rightarrow \nabla_{\partial/\partial r} \tilde{e}_a = 0 \Rightarrow \tilde{A}_r \equiv 0.$$

Use polar coords (r, ϕ_α):

$$F_{r\phi} = 0 \Rightarrow 0 = \partial_r \tilde{A}_\phi - \partial_\phi \tilde{A}_r + [\tilde{A}_r, \tilde{A}_\phi]$$

$$\Rightarrow 0 = \partial_r \tilde{A}_\phi \Rightarrow \tilde{A}_\phi \text{ does not depend on } r.$$

$$A = A_{\phi^\alpha}(\phi)\, d\phi^\alpha$$

$$g_{Euc} = dr^2 + r^2 g_{S^{n-1}}, \quad |\tilde{A}|^2_{g_{Euc},H}(0) < \infty.$$

$$|A|^2_{g,H} = g^{ij}\langle A_i, A_j\rangle_H$$

$$\Rightarrow |\tilde{A}|^2_{g_{Euc},H} = |\tilde{A}_r|^2 + r^{-2}|\tilde{A}_\phi|^2_{g_{S^{n-1}}}$$

send $r \to 0$, must have $\tilde{A}_\phi \equiv 0$.

$$\Rightarrow \tilde{A} \equiv 0 \text{ in } \{\tilde{e}_i\}. \quad \square$$

Simplified problem: $\alpha \in \Omega^1(\mathbb{R}^2)$ smooth.
Polar coords: $x = r\cos\phi$
$y = r\sin\phi$
Suppose on $\mathbb{R}^2 \setminus \{0\}$, we have $\alpha = f(\phi)\, d\phi$. Show: $\alpha \equiv 0$.

Exterior Covariant Derivative

Define $d_\nabla : \Omega^p(M,E) \to \Omega^{p+1}(M,E)$
where $\Omega^p(M,E) = \Omega^p(M) \otimes \Gamma(M,E)$, by:
p-forms valued in E

(1) $d_\nabla(s\alpha) = \nabla s \wedge \alpha + s\, d\alpha$
$s \in \Gamma(E)$, $\alpha \in \Omega^p$

(2) $d_\nabla(s_1\alpha_1 + s_2\alpha_2) = d_\nabla(s_1\alpha_1) + d_\nabla(s_2\alpha_2)$

In components:

$$d_\nabla\left(\frac{1}{p!}S^\alpha{}_{i_1\cdots i_p}\,dx^{i_1}\wedge\cdots\wedge dx^{i_p}\otimes e_\alpha\right)$$
$$=\frac{1}{p!}\nabla_K S^\alpha{}_{i_1\cdots i_p}\,dx^K\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_p}\otimes e_\alpha$$

$\nabla_K S^\alpha{}_{i_1\cdots i_p}$: $\nabla = d + A$ only acts on bundle index "α"

e.g. $\nabla_K S^\alpha{}_{ij} = \partial_K S^\alpha{}_{ij} + A_K{}^\alpha{}_\beta S^\beta{}_{ij}$.

Some texts define: $F = d_\nabla^{\,2}$.

check: $d_\nabla d_\nabla s$

$$= d_\nabla(ds + As)$$
$$= d(ds + As) + A\wedge(ds + As)$$
$$= dA\,s - A\,ds + A\,ds + A\wedge A\,s$$
$$= (dA + A\wedge A)s \quad \Rightarrow \quad d_\nabla^{\,2} s = Fs.$$

Bianchi identity: $\boxed{d_\nabla F = 0.}$

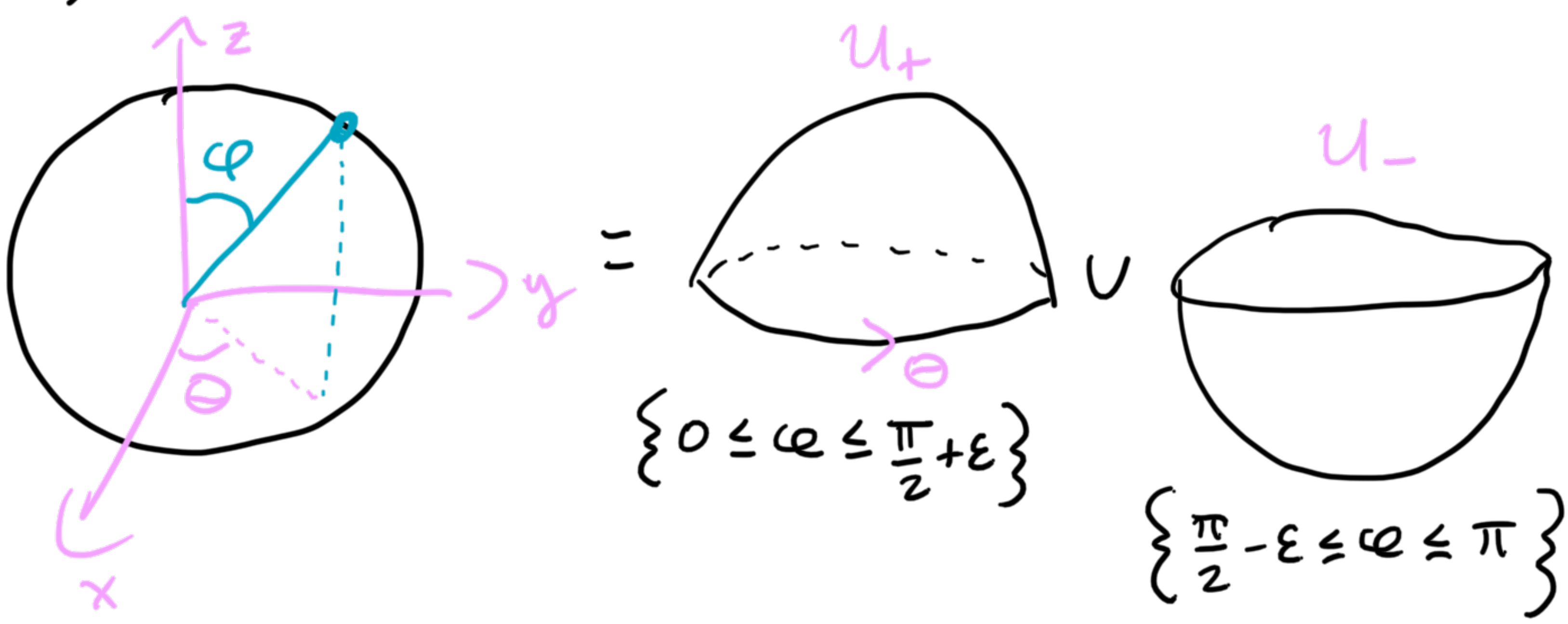
Here act d_∇ on $F \in \Omega^2(M, \mathrm{End}\,E)$

$$d_\nabla F = \frac{1}{2}\left(\partial_K F_{ij}{}^\alpha{}_\beta + A_K{}^\alpha{}_\gamma F_{ij}{}^\gamma{}_\beta - F_{ij}{}^\alpha{}_\gamma A_K{}^\gamma{}_\beta\right)dx^K\wedge dx^i\wedge dx^j$$
$$d_\nabla F = dF + A\wedge F - F\wedge A.$$

Expand:

$$d_\nabla F = d(dA + A\wedge A) + A\wedge(dA + A\wedge A) - (dA + A\wedge A)\wedge A$$
$$= dA\wedge A - A\wedge dA + A\wedge dA + A\wedge A\wedge A - dA\wedge A - A\wedge A\wedge A$$
$$= 0.$$

ex) S^2 with spherical coords



Define $L \to S^2$ by transition function

$$c_{+-} : \mathcal{U}_+ \cap \mathcal{U}_- \to U(1),$$

$$c_{+-} = \exp(2gi\theta), \quad 2g \in \mathbb{Z}.$$

c_{+-} well-defn when $\theta \to 2\pi$.

Define connection:

$$A_+ = ig(1 - \cos\varphi)\,d\theta$$

$$A_- = -ig(1 + \cos\varphi)\,d\theta$$

$$A_+ = c_{+-} A_- c_{+-}^{-1} - dc_{+-}\, c_{+-}^{-1}$$

$$A_+ = A_- - i2g\,d\theta$$

Field strength is $F = dA$.

$$F = ig \sin\varphi \, d\varphi \wedge d\theta.$$

"Magnetic flux": $\frac{1}{2\pi i}\int_{S^2} F = 2g \in \mathbb{Z}$.