Connections on TM

- In this section, we restrict to connections on $E = TM$. Here is a summary of the formulas from the earlier lecture:

- Notation:
  - g_{ij} metric on TM
  - ∇ metric compatible connection
  - $\nabla_{\partial_i} g(V, W) = g(\nabla_{\partial_i} V, W) + g(V, \nabla_{\partial_i} W)$
  - $V = V^i \dfrac{\partial}{\partial x^i}$ vector field
  - ∂_i local coordinate frame

$$\nabla_{\partial_i} \partial_j = \Gamma_{ij}^{\ \ k} \partial_k$$

$A_i{}^k{}_j = \Gamma_{ij}^{\ \ k}$ TM notation in coord frame

$$\nabla_{\partial_i} V = (\nabla_i V^k) \partial_k$$

$$\nabla_i V^k = \partial_i V^k + \Gamma_{ij}^{\ \ k} V^j$$

Lee writes $V^k{}_{;p}$ for $\nabla_p V^k$

$$\nabla_W V = W^i \nabla_{\partial_i} V$$

Transformation law: $c_{uv}{}^i{}_j = \dfrac{\partial x_u^i}{\partial x_v^j}$,

$$(\Gamma_u)_{ij}^{\ \ k} = \frac{\partial x_u^k}{\partial x_v^p} (\Gamma_v)_{rs}^{\ \ p} \frac{\partial x_v^r}{\partial x_u^i} \frac{\partial x_v^s}{\partial x_u^j} - \frac{\partial x_v^r}{\partial x_u^i} \frac{\partial x_v^s}{\partial x_u^j} \frac{\partial^2 x_u^k}{\partial x_v^r \partial x_v^s}. \quad (*)$$

Along curves: $V = V^i(t) \partial_i|_{\gamma(t)}$ along $\gamma(t)$

$$D_t V^k = \partial_t V^k + \dot\gamma^i \Gamma_{ij}^{\ \ k}(\gamma(t)) V^j$$

$$\partial_t \langle V, W \rangle_g = \langle D_t V, W \rangle_g + \langle V, D_t W \rangle_g$$

Torsion: No such concept on general $E \to M$

Torsion of ∇: $T \in \Gamma(M, TM \otimes \Omega^2)$

$$T = \frac{1}{2} T_{ij}^{\;\;k} \, dx^i \wedge dx^j \otimes \frac{\partial}{\partial x^k}$$

$$T_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k .$$

Check: T satisfies transformation law for $TM \otimes \Omega^2$.

Can define torsion without using coords:

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] .$$

Recall: $[X, Y] = (X^i \partial_i Y^k - Y^i \partial_i X^k) \partial_k .$

Need to check: $T(X, Y)$ consistent with:

$X^i Y^j T_{ij}^{\;\;k} \partial_k$, where $X = X^i \partial_i$, $Y = Y^j \partial_j$
$T_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k .$

Say ∇ is torsion-free if:

$$T \equiv 0 \iff \nabla_X Y - \nabla_Y X = [X, Y]$$
$$\iff \Gamma_{ij}^k = \Gamma_{ji}^k$$

Levi-Civita Connection

Thm: Let (M, g) be Riemannian mfd.

There is a unique connection ∇ on TM s.t.

(1) ∇ is metric compatible
(2) ∇ is torsion-free

<u>Note</u>: This connection is called the Levi-Civita connection and from now on we reserve the notation

$$\nabla_i X^k = \partial_i X^k + \Gamma_{ij}^{\ \ k} X^j$$ for this connection.

<u>Pf</u>: Start with uniqueness. Suppose such a connection exists. Then:

$$(1)\quad 0 = \nabla_i g_{kj} = \partial_i g_{kj} - \Gamma_{ik}^{\ \ l} g_{lj} - \Gamma_{ij}^{\ \ l} g_{kl}$$

$$(2)\quad \Gamma_{ij}^{\ \ k} = \Gamma_{ji}^{\ \ k}$$

$$\Downarrow$$

$$\partial_i g_{kj} = \Gamma_{ik}^{\ \ l} g_{lj} + \Gamma_{ij}^{\ \ l} g_{kl}$$

Consider:

$$\partial_i g_{kj} + \partial_j g_{ki} - \partial_k g_{ij}$$

$$= \left(\cancel{\Gamma_{ik}^{\ \ l} g_{lj}} + \Gamma_{ij}^{\ \ l} g_{kl}\right) + \left(\cancel{\Gamma_{jk}^{\ \ l} g_{li}} + \Gamma_{ji}^{\ \ l} g_{kl}\right)$$

$$- \left(\cancel{\Gamma_{ki}^{\ \ l} g_{lj}} + \cancel{\Gamma_{kj}^{\ \ l} g_{il}}\right)$$

$$\Rightarrow g_{lk} \Gamma_{ij}^{\ \ l} = \frac{1}{2}\left(\partial_i g_{kj} + \partial_j g_{ki} - \partial_k g_{ij}\right)$$

$$\Rightarrow \Gamma_{ij}^{\ \ k} = \frac{g^{kl}}{2}\left(-\partial_l g_{ij} + \partial_i g_{lj} + \partial_j g_{il}\right) \qquad (*)$$

Metric uniquely determines Levi-Civita ✓

Conversely, define $\nabla = d + \Gamma$ by $(*)$.

Homework: check
- well-defined connection
- $\nabla_i g_{kj} = 0$
- $\Gamma_{ij}^{\ \ k} = \Gamma_{ji}^{\ \ k}$

More details about well-defn: this means

$$W^{k}{}_{i} := \nabla_i V^k \text{ defines } W \in \Gamma(M, T^*M \otimes TM).$$

On overlaps of coords $(\mathcal{U}, x^i)$, $(\tilde{\mathcal{U}}, \tilde{x}^\mu)$ need

$$\tilde{\nabla}_\mu \tilde{V}^\nu = \frac{\partial \tilde{x}^\nu}{\partial x^i} \frac{\partial x^k}{\partial \tilde{x}^\mu} \nabla_i V^k, \quad \text{where}$$

$$\tilde{\nabla}_\mu \tilde{V}^\nu = \frac{\partial}{\partial \tilde{x}^\mu} \tilde{V}^\nu + \tilde{\Gamma}^\nu_{\mu\rho} \tilde{V}^\rho$$

$$\tilde{V}^\mu = \frac{\partial \tilde{x}^\mu}{\partial x^i} V^i.$$

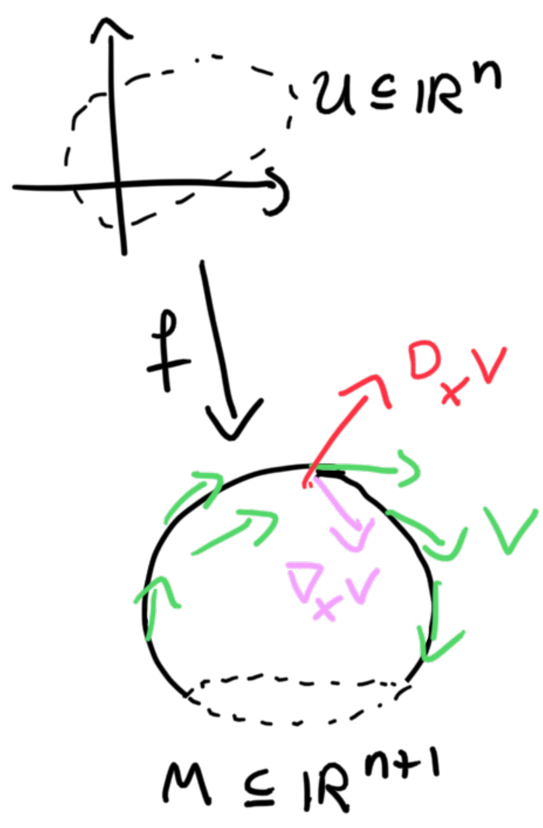
Exercise: check this using $(*)$.

There is also a formula without indices:

$$g(\nabla_X Y, Z) = \frac{1}{2}\Big[X g(Y, Z) + Y g(Z, X) - Z g(X, Y) + g(Z, [X, Y]) + g(Y, [Z, X]) + g(X, [Z, Y]) \Big]$$

(check consistent with $(*)$)

ex) $f : \mathcal{U} \to M \subset \mathbb{R}^{n+1}$ (coords u^i).
hypersurface with normal ν.



Vector fields on M:

$$V = V^i(u) \frac{\partial f}{\partial u^i} \in \Gamma(TM)$$

Pull back metric $g = f^* g_{Euc}$

$$g_{ij} = \left\langle \frac{\partial f}{\partial u^i}, \frac{\partial f}{\partial u^j} \right\rangle_{g_{Euc}}.$$

In this setup, the Levi-Civita connection turns out to be: for $X, V \in \Gamma(TM)$

$$\nabla_X V = D_X V - \langle D_X V, \nu \rangle_{g_{EUC}} \nu$$

$D_X V$ directional derivative in $\mathbb{R}^{n+1}$

or:

$$\nabla_X V = (D_X V)^T$$ projection onto tangent space

Later in notes: we prove (without using indices) that ∇ is metric compatible and torsion-free $\Rightarrow \nabla$ is Levi-Civita.

Here: alternate direct computation that ∇ is LC:

Write $f_i = \frac{\partial f}{\partial u^i}$ for local frame of sections

$$\nabla_{\frac{\partial}{\partial u^i}} f_j = \Gamma_{ij}^k f_k$$ by defn of connection coeff. Will solve for Γ_{ij}^k.

$$\nabla_{\partial/\partial u^i} f_j = \left(\frac{\partial}{\partial u^i} f_j\right)^T = (f_{ij})^T.$$

$$\Rightarrow \Gamma_{ij}^k g_{k\ell} = \langle f_{ij}, f_\ell \rangle$$ since $g_{k\ell} = \langle f_k, f_\ell \rangle$.

Notice:

$$\frac{1}{2}(-\partial_\ell g_{ij} + \partial_i g_{\ell j} + \partial_j g_{\ell i})$$
$$= \frac{1}{2}\left(-\partial_\ell \langle f_i, f_j \rangle + \partial_i \langle f_\ell, f_j \rangle + \partial_j \langle f_\ell, f_i \rangle\right)$$
$$= \frac{1}{2}\left(2 \langle f_{ij}, f_\ell \rangle\right)$$ after cancellation.

$$\Rightarrow \Gamma_{ij}^k g_{k\ell} = \frac{1}{2}(-\partial_\ell g_{ij} + \partial_i g_{\ell j} + \partial_j g_{\ell i})$$

$\Rightarrow \Gamma_{ij}^k$ agrees with the formula for the Levi-Civita connection.

Note on torsion: Let $H \in \Omega^3(M)$

Can encode in geometry by defining:

$$\tilde{\nabla} = \nabla^{LC} + \frac{g^{-1}H}{2}$$

$$\tilde{\nabla}_i V^k = \nabla_i^{LC} V^k + \frac{H}{2}{}^k{}_{ij} V^j, \quad H^k{}_{ij} = g^{kl} H_{lij}$$

$$\Rightarrow \tilde{\Gamma}_{ij}^k = (\Gamma^{LC})_{ij}^k + \frac{H}{2}{}^k{}_{ij}$$

Torsion: $\tilde{T}_{ij}^k = \tilde{\Gamma}_{ij}^k - \tilde{\Gamma}_{ji}^k$

$$= \frac{1}{2}\left(H^k{}_{ij} - H^k{}_{ji}\right)$$

$$\Rightarrow \tilde{T}_{ij}^k = H^k{}_{ij}.$$

Torsion creates "flux" on mfd without boundary:

$$\int_M (\tilde{\nabla}_i V^i)\, d\mathrm{vol}_g = \frac{1}{2}\int_M (H^i{}_{ij} V^j)\, d\mathrm{vol}_g$$

For Levi-Civita: divergence Thm $\int_M (\nabla_i V^i)\, d\mathrm{vol}_g = 0.$

will prove later