Covariant Derivatives

A covariant derivative (or connection) on a bundle $E \to M$ is a map

$$\nabla : \Gamma(M,E) \to \Gamma(M, E \otimes T^*M) \quad \text{s.t.}$$

(1) $\nabla(as + bs') = a\nabla s + b\nabla s'$

(2) $\nabla(fs) = f\nabla s + df \otimes s$

$a, b \in \mathbb{R}$
$s, s' \in \Gamma(M,E)$
$f \in C^\infty(M,\mathbb{R})$

(Section of T^*M means: $\nabla_{x^i\partial_i + y^i\partial_i} s = x^i \nabla_{\partial_i} s + y^i \nabla_{\partial_i} s$)
with x^i, y^i component functions

Local Expression:

In local frame $\{e_\alpha\}_{i=1}^{k}$ over $\mathcal{U} \subseteq M$, denote:

$$\nabla_{\partial_i} e_\alpha = A_i{}^\beta{}_\alpha \, e_\beta \qquad \text{"connection coefficients"}$$

$\nabla_{\partial_i} = \nabla_{\frac{\partial}{\partial x^i}}$, x^i coords on $\mathcal{U}$

Another common notation: $\nabla_{\partial_i} e_\alpha = \omega_i{}^\beta{}_\alpha \, e_\beta$.

Write $s = s^\alpha e_\alpha$, then

$$\begin{aligned}\nabla_{\partial_i} s &= \nabla_{\partial_i}(s^\alpha e_\alpha) \\ &= \partial_i s^\alpha e_\alpha + s^\alpha \nabla_{\partial_i} e_\alpha \\ &= (\partial_i s^\alpha + A_i{}^\alpha{}_\mu s^\mu)\, e_\alpha .\end{aligned}$$

Notation: $\nabla_{\partial_i} s = (\nabla_i s^\alpha) e_\alpha$,

$$\nabla_i s^\alpha = \partial_i s^\alpha + A_i{}^\alpha{}_\beta s^\beta$$

⚠ Some books write $(\nabla_i s)^\alpha$ instead of $\nabla_i s^\alpha$.

<u>Notation:</u> write $\nabla = d + A$, where $A \overset{loc}{=} A_i{}^\alpha{}_\beta \, dx^i$
"gauge potential"

Transformation law for A:

Need $\nabla_i s^\alpha$ to be a section:

$$\nabla_i s_u^\alpha = c_{uv}{}^\alpha{}_\beta \nabla_i s_v^\beta .$$

$\nabla_i s_u$ uses A_u
$\nabla_i s_v$ uses A_v

Expand:

$$\partial_i s_u^\alpha + (A_u)_i{}^\alpha{}_\beta s_u^\beta = c_{uv}{}^\alpha{}_\beta \left(\partial_i s_v^\beta + (A_v)_i{}^\beta{}_\gamma s_v^\gamma\right)$$

Transform $s_u \mapsto s_v$:

$$\partial_i c_{uv}{}^\alpha{}_\beta s_v^\beta + c_{uv}{}^\alpha{}_\beta \partial_i s_v^\beta + (A_u)_i{}^\alpha{}_\beta c_{uv}{}^\beta{}_\gamma s_v^\gamma = \text{RHS}$$

$$\Rightarrow \partial_i c_{uv}{}^\alpha{}_\gamma s_v^\gamma + (A_u)_i{}^\alpha{}_\beta c_{uv}{}^\beta{}_\gamma s_v^\gamma = c_{uv}{}^\alpha{}_\beta (A_v)_i{}^\beta{}_\gamma s_v^\gamma$$

$$(A_u)_i = c_{uv}(A_v)_i c_{uv}^{-1} - \partial_i c_{uv} \, c_{uv}^{-1} \qquad (*)$$

Conversely: given $\{(\mathcal{U}, A_u)\}$ satisfying $(*)$ we obtain a covariant derivative $\nabla = d + A$

<u>Summary:</u> if $s \mapsto Qs$, then

$$\nabla_i s \mapsto Q \nabla_i s$$

$$A \mapsto QAQ^{-1} - dQ\,Q^{-1}$$

transformation law for gauge potential

Inducing Covariant Derivatives

<u>Note:</u> on scalar function $f \in C^\infty(M, \mathbb{R})$, denote

$$\nabla_i f = \frac{\partial f}{\partial x^i} .$$

Given ∇ on E, can induce ∇ on E^* by:

$$\nabla_i (s^\alpha \phi_\alpha) = \nabla_i s^\alpha \phi_\alpha + s^\alpha \nabla_i \phi_\alpha$$

$s \in \Gamma(M, E)$
$\phi \in \Gamma(M, E^*)$

impose the product rule

Product rule implies:

$$\partial_i (\cancel{s^\alpha} \phi_\alpha) = \cancel{\partial_i s^\alpha} \phi_\alpha + A_i{}^\alpha{}_\gamma s^\gamma \phi_\alpha + s^\alpha \nabla_i \phi_\alpha$$

$$\Rightarrow s^\alpha (\partial_i \phi_\alpha - A_i{}^\alpha{}_\gamma \phi_\alpha) = s^\alpha \nabla_i \phi_\alpha$$

Define ∇ on E^* by:

$$\nabla_i \phi_\alpha = \partial_i \phi_\alpha - \phi_\beta A_i{}^\beta{}_\alpha .$$

Can define ∇ on $E^{\otimes p} \otimes (E^*)^{\otimes q}$ similarly.

ex) $h \in \Gamma(M, E^* \otimes E^*)$

$$\nabla_i h_{\alpha\beta} = \partial_i h_{\alpha\beta} - h_{\mu\beta} A_i{}^\mu{}_\alpha - h_{\alpha\mu} A_i{}^\mu{}_\beta$$

ex) $s \in \Gamma(M, \operatorname{End} E)$

$$\nabla_i s^\alpha{}_\beta = \partial_i s^\alpha{}_\beta + A_i{}^\alpha{}_\mu s^\mu{}_\beta - s^\alpha{}_\mu A_i{}^\mu{}_\beta$$

$$\nabla_i s = \partial_i s + A s - s A$$

$$\nabla s = ds + [A, s].$$

matrix multiplication implied

$$[s^\alpha{}_\beta] = \begin{pmatrix} s^1{}_1 & s^1{}_2 \\ s^2{}_1 & s^2{}_2 \end{pmatrix}$$

Can check:

(1) induced ∇ transforms correctly
e.g. $\nabla_i \phi^u = \nabla_i \phi^v c_{vu}$ $\phi \in \Gamma(M, E^*)$

(2) higher product rules hold (*)
e.g. $\partial_i (h_{\alpha\beta} s^\alpha \psi^\beta) = \nabla_i h_{\alpha\beta} s^\alpha \psi^\beta + h_{\alpha\beta} \nabla_i s^\alpha \psi^\beta + h_{\alpha\beta} s^\alpha \nabla_i \psi^\beta.$

Metric Connection:

Let H be metric on $E \to M$.

∇ is metric compatible connection if:

$s, \psi \in \Gamma(M, E)$

$$\partial_i \langle s, \psi \rangle_H = \langle \nabla_{\partial_i} s, \psi \rangle_H + \langle s, \nabla_{\partial_i} \psi \rangle_H$$

In components:

$$\partial_i (H_{\alpha\beta} s^\alpha \psi^\beta) = H_{\alpha\beta} \nabla_i s^\alpha \psi^\beta + H_{\alpha\beta} s^\alpha \nabla_i \psi^\beta$$

By (*), ∇ metric comp $\iff$ $\nabla_i H_{\alpha\beta} = 0$.

In full components:

$$\partial_i H_{\alpha\beta} = H_{\mu\beta} A_i{}^\mu{}_\alpha + H_{\alpha\mu} A_i{}^\mu{}_\beta .$$

Parallel Transport:

- $\nabla = d + A$ on $E \to M$
- $\gamma : [0,1] \to M$ path from p to q.
- $\gamma^* E \to [0,1]$ pullback bundle has induced derivative

$$D_t s^\alpha(t) = \partial_t s^\alpha(t) + \dot{\gamma}^i A_i{}^\alpha{}_\beta(\gamma(t)) s^\beta(t)$$

Notation: $D_t = \gamma^* \nabla$, $f^* \nabla = d + f^* A$

$$s \in \Gamma([0,1], \gamma^* E), \quad s \overset{\text{loc}}{=} \begin{bmatrix} s^1(t) \\ \vdots \\ s^k(t) \end{bmatrix}$$

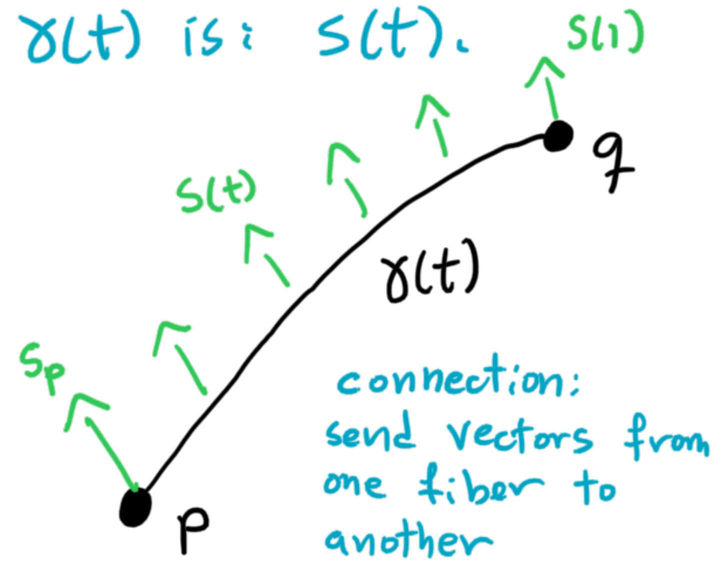
<u>Can check</u>: $D_t s$ transforms as section of $\gamma^* E$.
trans fun of $\gamma^* E$ are $c_{uv}(\gamma(t))$

<u>Def</u>: s is parallel along γ if $D_t s \equiv 0$.

$\gamma : [0,1] \to M$

<u>Prop</u>: Given ∇, $s_p \in E|_p$, $\exists !\ s(t) \in \Gamma([0,1], \gamma^* E)$ s.t. $D_t s \equiv 0$ and $s(0) = s_p$.

Parallel transport of s_p along $\gamma(t)$ is: $s(t)$.



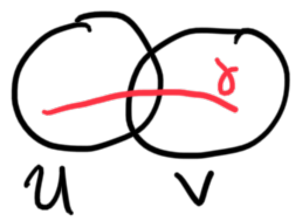
<u>Pf</u>: In local triv, need

$$0 = \dot{s}^\alpha + \dot{\gamma}^i A_i{}^\alpha{}_\beta(\gamma) s^\beta .$$

Linear ODE for $s(t)$.

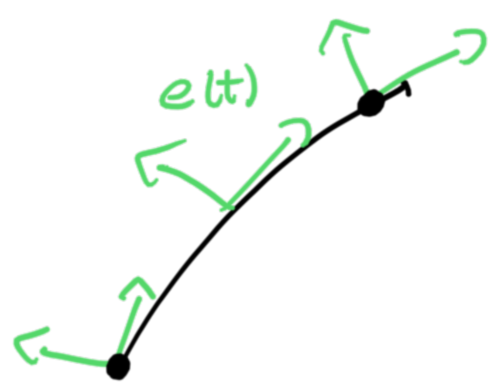
Unique soln given initial cond.

If crosses trivializations:



stop on overlap, restart in next triv. $\square$

<u>Note</u>: If $\{e_1, \dots, e_K\}$ ONB at $E|_p$, ∇ metric connection, then parallel transport $\{e_1(t), \dots, e_K(t)\}$ ONB at all $E|_{\gamma(t)}$.



$$\frac{d}{dt} \langle e_i(t), e_j(t) \rangle$$
$$= \langle D_t e_i, e_j \rangle + \langle e_i, D_t e_j \rangle$$
$$= 0.$$ Angles constant + $|e_i(t)|^2 \equiv 1$.

Here we used: $s, \psi \in \Gamma([0,1], \gamma^* E)$, ∇ metric connection

$$\frac{d}{dt} \langle s, \psi \rangle_H = \langle D_t s, \psi \rangle_H + \langle s, D_t \psi \rangle_H$$

Proof below:

Expand LHS:

$$\partial_t \left(H_{\alpha\beta}(\gamma(t))\, s^\alpha(t)\, \psi^\beta(t) \right)$$

$$= \underbrace{\partial_i H_{\alpha\beta}\, \dot\gamma^i s^\alpha \psi^\beta}_{} + H_{\alpha\beta}\, \dot s^\alpha \psi^\beta + H_{\alpha\beta}\, s^\alpha \dot\psi^\beta$$

$$\left(A_i{}^\mu{}_\alpha H_{\mu\beta} + A_i{}^\mu{}_\beta H_{\alpha\mu} \right) \dot\gamma^i s^\alpha \psi^\beta$$

metric connection identity for $\partial_i H_{\alpha\beta}$

relabel indices

$$= H_{\alpha\beta} \left(\dot s^\alpha + \dot\gamma^i A_i{}^\alpha{}_\nu s^\nu \right) \psi^\beta$$

$$+ H_{\alpha\beta}\, s^\alpha \left(\dot\psi^\beta + \dot\gamma^i A_i{}^\beta{}_\nu \psi^\nu \right)$$