Laplace Comparison: (ref: Walpuski, Lee)

- Let $p \in M$. Let $\{x^i\}$ normal coords at p.
- $r = \sqrt{\sum (x^i)^2}$ radial distance to p.

First computation of Δr:

$$\Delta r = \frac{1}{\sqrt{g}} \partial_i \Big[\sqrt{g} \, \underbrace{g^{ij} \partial_j r}_{= \frac{x^i}{r} \text{ by Gauss lemma}} \Big]$$

$$= \partial_i \left(\frac{x^i}{r} \right) + \frac{1}{\sqrt{g}} \frac{x^i}{r} \partial_i \sqrt{g}$$

$$\Delta r = \frac{n-1}{r} + O(1) \text{ on } B_{\mathrm{inj}(p)}(0).$$

$$S_c(\rho) = \begin{cases} \frac{1}{\sqrt{c}} \sin \sqrt{c} \rho & c > 0 \\ \rho & c = 0 \\ \frac{1}{\sqrt{c}} \sinh \sqrt{c} \rho & c < 0 \end{cases}$$

$\begin{cases} A_{ij} \geq B_{ij} \text{ as matrices means:} \\ A_{ij} v^i v^j \geq B_{ij} v^i v^j \ \forall v \end{cases}$

Thm: Suppose $R_{ij} \geq (n-1) c \, g_{ij}$, for $c \in \mathbb{R}$.
Let $p \in M$, $\mathcal{U}$ normal nbhd of p, r distance to p.
Then on $\mathcal{U}$:

$$\Delta r \leq (n-1) \frac{s_c'(r)}{s_c(r)}.$$

(if $c > 0$, only holds on $\{ r < \frac{\pi}{\sqrt{c}} \} \cap \mathcal{U}$)

As a corollary, we obtain Myers's thm.

Thm: (Myers) Let (M, g) be complete and connected, $R_{ij} \geq (n-1) c \, g_{ij}$ with $c > 0$.

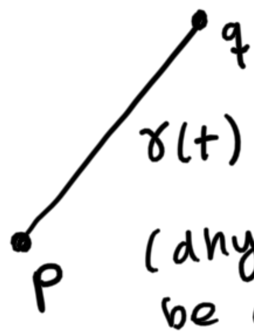
Then:
- $\mathrm{diam}(M, g) \leq \frac{\pi}{\sqrt{c}}$.
- M is compact

- Fundamental group of M is finite, i.e. $b_1(M) = 0$.

Pf of Myers (assuming Laplace comp)

(1) Diam bdd:

Suppose $\operatorname{diam}(M,g) > \frac{\pi}{\sqrt{c}}$.



$\exists$ minimizing unit speed geodesic from p to q

(any 2 pts can be connected by minimizing geodesic since (M,g) complete)

$\gamma : [0,T] \to M$ with $T > \frac{\pi}{\sqrt{c}}$.

$\gamma(t) = \exp_p(tv)$.

$\Rightarrow$ no conj pts along $\gamma(t)$.

$\Rightarrow$ $\exp_p$ local diffeo along $\gamma(t)$

$\Rightarrow \exists W \supseteq \gamma(t)$ with $\exp_p^* g = \tilde{g}$ not degenerate.

Laplace comp on $(W, \tilde{g})$ (normal nbhd of $0 \in W$)

$$\underbrace{\Delta r \circ \gamma(t)}_{O(1) \text{ as } t \to \pi/\sqrt{c}} \leq \underbrace{(n-1)\frac{s_c'(t)}{s_c(t)}}_{\to -\infty \text{ as } t \to \pi/\sqrt{c}}, \quad s_c = \frac{1}{\sqrt{c}}\sin(\sqrt{c}\,t)$$

Contradiction.

Note: earlier computation in normal coords

$\Delta r = \frac{n-1}{r} + O(1)$ on $B_{\frac{\pi}{\sqrt{c}}}(0)$.

$r \circ \gamma(t) = t \Rightarrow \Delta r \circ \gamma(t) = \frac{n-1}{t} + O(1)$.

problem at $t = 0$, not $t = \frac{\pi}{\sqrt{c}}$.

(2) Compactness:

Follows from $\operatorname{diam}(M,g) \leq D$.

Have surjection $\exp_p : \overline{B_D}(0) \to M$.

(3) Fundamental group:

$\pi : \tilde{M} \to M$ universal cover.

$\tilde{g} = \pi^* g$.

$(\tilde{M}, \tilde{g})$ complete with Ricci lower bdd.

$\Rightarrow \tilde{M}$ compact

$\Rightarrow \pi_1(M)$ is finite. $\square$

We now start the proof of Laplace comp.

First, recall Bochner formula (HWK)

$$\Delta |\nabla f|^2 = 2|\nabla^2 f|^2 + 2\langle \nabla \Delta f, \nabla f\rangle + 2R^{pq}\partial_p f \partial_q f,$$

for all $f \in C^\infty(M, \mathbb{R})$.

Apply to r with:

$|\nabla r| = 1, \quad R_{ij} \geq c(n-1) g_{ij}$

$$\Rightarrow 0 \geq |\nabla^2 r|^2 + \langle \nabla \Delta r, \nabla r\rangle + c(n-1).$$

Gauss Lemma $\rightarrow$ $\underbrace{g^{ij}\partial_j r}_{= x^i/r} \partial_i (\Delta r)$, $\quad \frac{x^i}{r}\partial_i := \partial_r$

$$\Rightarrow |\nabla^2 r|_g^2 + \partial_r \Delta r + c(n-1) \leq 0$$

Let $A^i{}_j = \nabla^i \nabla_j r$.

$|\nabla^2 r|^2 = \sum (A^i{}_j)^2 = \sum_{i=1}^n \lambda_i^2$ eigenvalues of A.

Note A has a zero eigenvalue: $(\nabla^2 r)(\partial_r, \partial_r) = 0$.

$v^i = x^i, \quad v^T A v = \sum x^i A^i{}_j x^j$

$$= \underbrace{x^i g^{ik}}_{\text{Gauss lemma}} \big(\underbrace{\partial_k \partial_j r - \Gamma_{kj}^l \partial_l r}_{\nabla_k \nabla_j r}\big) x^j$$

$$= \underbrace{x^k x^j \partial_k \partial_j r}_{=0 \text{ direct comp}} - \underbrace{\Gamma_{kj}^l x^k x^j}_{=0 \text{ normal coords}} \partial_l r$$

$$= 0.$$

$$\Rightarrow |\nabla^2 r|_g^2 = \sum_{i=1}^{n-1} \lambda_i^2 \quad \text{non-zero eigenvalues}$$

$$\geq \frac{1}{n-1}\left(\sum_{i=1}^{n-1} \lambda_i\right)^2 \quad \text{Cauchy-Schwarz}$$

$$= \frac{1}{n-1}(\Delta r)^2 \quad \Delta r = \operatorname{Tr} A = \nabla^i \nabla_i r.$$

$$\Rightarrow \frac{1}{n-1}(\Delta r)^2 + \partial_r(\Delta r) + c(n-1) \leq 0.$$

Let $\gamma(t)$ be a radial unit speed geodesic starting at p.

Let $f(t) = \frac{\Delta r}{n-1} \circ \gamma(t)$. $\quad \gamma(t) = t(v^1, \ldots, v^n)$, $r \circ \gamma(t) = t$

$$\partial_r = \frac{x^i}{r} \partial_i$$

$$\partial_r \big|_{\gamma(t)} = \frac{\cancel{t} v^i}{\cancel{t}} \partial_i$$

$$f'(t) = \frac{1}{n-1} \partial_i(\Delta r)\dot{\gamma}^i$$

$$= \frac{1}{n-1} \partial_r(\Delta r) \circ \gamma(t).$$

$$\Rightarrow f' + f^2 + c \leq 0.$$ Also, $f(t) = \frac{1}{t} + O(1)$.
(computed at top of notes)

Riccati comparison (below) $\Rightarrow f(t) \leq \frac{s_c'(t)}{s_c(t)}$.

$$\Rightarrow \Delta r \leq (n-1) \frac{s_c'(r)}{s_c(r)}$$ evaluated along $\gamma(t)$.

Since $\gamma(t)$ arbitrary $\Rightarrow$ Laplace comparison. $\square$

<u>Riccati comparison</u> : $f : (0, R) \to \mathbb{R}$
arbitrary,

- $f' + f^2 + c \le 0$
- $f(t) = \frac{1}{t} + O(1)$

$$\Rightarrow f(t) \le \frac{s_c'(t)}{s_c(t)}.$$

<u>Pf</u>: Note $H = s_c'(t)/s_c(t)$ solves

$$s_c(t) = \begin{cases} \frac{1}{\sqrt{c}} \sin\sqrt{c}\,t & \rightsquigarrow H = \sqrt{c}\,\frac{\cos\sqrt{c}t}{\sin\sqrt{c}t} \\ t & \rightsquigarrow H = \frac{1}{t} \\ \frac{1}{\sqrt{c}} \sinh\sqrt{c}\,t & \rightsquigarrow \ldots \end{cases}$$

$$H' + H^2 + c = 0$$
$$H = \frac{1}{t} + O(1)$$

Consider $G(t) = -\int_t^R (f(s) + H(s))\,ds$ so that

$$G' = f + H$$
$$G = 2\log t + O(1).$$

Then:

$$\frac{d}{dt}\left(e^G(f - H)\right) = e^G(f+H)(f-H) + e^G(f' - H')$$
$$= e^G\left[\underline{f^2} - H^2 + \underline{f'} - H' + \underline{c} - c\right]$$
$$\le 0.$$

$$\Rightarrow e^G(f-H) \searrow$$

$$\Rightarrow e^G(f-H) \le \lim_{t\to 0} \underbrace{e^{G(t)}}_{e^{2\log t}\, e^{O(1)}} \underbrace{(f(t) - H(t))}_{O(1)} \to 0.$$

$$\Rightarrow f(t) \le H(t). \quad \square$$