# Gauss Lemma:

- Let $p \in M$, $\{x^i\}$ normal coords. Then:

$$\boxed{g_{ij}(x)\,x^j = \delta_{ij}\,x^j.}$$

Pf: Let $v \in \mathbb{R}^n$ unit vector.

$\gamma(t) = t\begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}$ is geodesic in normal coords.

We will show: $\frac{d}{dt}\left[\left(\underbrace{g_{ij}x^j - \delta_{ij}x^j}_{=0 \text{ at } t=0}\right)(\gamma(t))\right] = 0.$

$\Rightarrow \left(g_{ij}(x)x^j - \delta_{ij}x^j\right)(tv) \equiv 0.$ (can write any pt as $x = tv$)

By chain rule, we must show

$\partial_p\left(g_{kj}x^j - x^k\right)x^p = 0,$

$$\frac{d}{dt} f(\overbrace{\gamma(t)}^{tv}) = \partial_p f\,\dot{\gamma}^p = \partial_p f\, v^p = \partial_p f\,\frac{x^p}{t}$$

1. $\gamma(t) = tv$ geodesic eqn

$\Gamma_{ij}^k(tv)(tv)^i(tv)^j = 0$ $\quad x = tv$

$\Gamma_{ij}^k(x)\,x^i x^j = 0 \quad (*)$

2. Geodesics have const speed

normal coords $g_{ij}(0) = id$

$g_{ij}(\gamma(t))v^iv^j = g_{ij}(0)v^iv^j = \delta_{ij}v^iv^j.$

$g_{ij}(x)x^ix^j = \delta_{ij}x^ix^j \quad (**)$

Expand $(*)$:

$$g_{pk}g^{kl}\left(-\partial_l g_{ij} + \overbrace{\partial_i g_{lj} + \partial_j g_{il}}^{2\partial_i g_{lj}x^ix^j}\right)x^ix^j = 0.$$

$$\Rightarrow \partial_p g_{ij} x^i x^j = 2\, \partial_i g_{pj} x^i x^j$$

$$\Rightarrow \partial_p (g_{ij} x^i x^j) - 2 g_{pj} x^j = 2\, \partial_i (g_{pj} x^j) x^i - 2 g_{pj} x^j$$

$$\Rightarrow \partial_p (\underbrace{g_{ij} x^i x^j}_{\delta_{ij} x^i x^j \text{ by } (**)}) = 2\, \partial_i (g_{pj} x^j) x^i$$

$$\Rightarrow \partial_p (\Sigma (x^i)^2) = 2\, \partial_i (g_{pj} x^j) x^i$$

$$\partial_p x^i x^i = \partial_i (g_{pj} x^j) x^i$$

can switch

$$\Rightarrow \partial_i (g_{pj} x^j - x^p) x^i = 0 \quad \text{as required.} \quad \square$$

Cor: Introduce notation:

- $r := (\Sigma (x^i)^2)^{1/2}$
- $\frac{\partial}{\partial r} := \frac{x^i}{r} \frac{\partial}{\partial x^i}$

The gradient of $r$:

- $\nabla r = (g^{ik} \partial_k r) \frac{\partial}{\partial x^i}$, and note: $\partial_i r = \frac{x^i}{r}$.

Gauss lemma implies:

- $g^{ik} \partial_k r = x^i / r \quad \Rightarrow \nabla r = \frac{\partial}{\partial r}$
- $|\nabla r|_g = 1$.

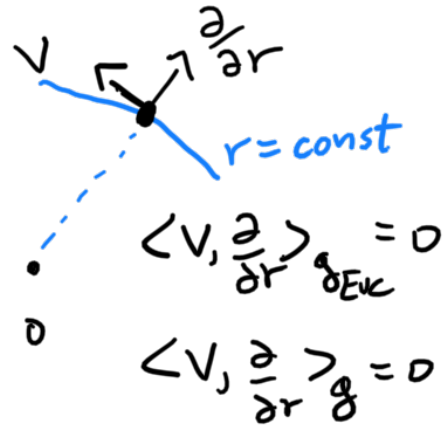
Pf:

$$g^{ik} \partial_k r = g^{ik} \frac{x^k}{r} \overset{\text{Gauss lemma}}{=} g^{ik} g_{kl} \frac{x^l}{r} = \frac{x^i}{r}.$$

$$|\nabla r|^2 = g^{ik} \partial_i r \partial_k r = \Sigma \frac{x^i}{r} \frac{x^i}{r} = 1. \quad \square$$

**Cor:** $g = dr^2 + g^r$ ← metric on tangent space of spheres $\{r = \text{const}\}$

if $\langle V, \frac{\partial}{\partial r} \rangle_{g_{Euc}} = 0$, then $\langle V, \frac{\partial}{\partial r} \rangle_g = 0$.




**Pf:** Suppose $\delta_{ij} V^i \frac{x^j}{r} = 0$.

Then:

$$\langle V, \tfrac{\partial}{\partial r} \rangle_g = g_{ij} V^i \frac{x^j}{r} = \delta_{ij} V^i \frac{x^j}{r} = 0. \quad \square$$

**Def:** The length of a curve $\gamma : [a,b] \to M$ is

$$L(\gamma) = \int_a^b |\dot\gamma|_g \, dt, \quad |\dot\gamma| = (g_{ij} \dot\gamma^i \dot\gamma^j)^{1/2}$$

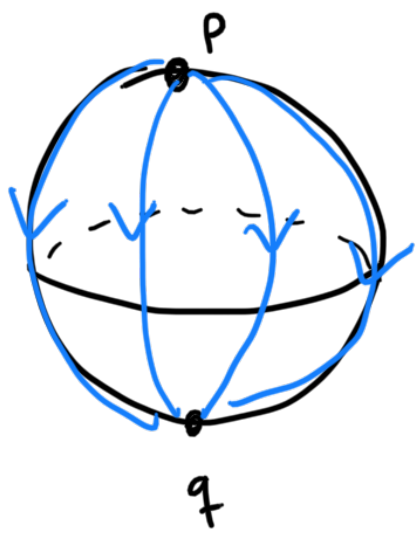
**Def:** $$d(p,q) = \inf_{\substack{\gamma \text{ path} \\ \text{from } p \text{ to } q}} L(\gamma).$$

← path is image of cont piecewise smooth map from interval to $M$

**Thm:** Let $p \in M$. Let $\mathcal{U} = \exp_p(B_\varepsilon(0))$ normal nbhd.

$\forall q \in \mathcal{U}$, there is a geodesic from $p$ to $q$ which is the unique length minimizing curve from $p$ to $q$.

ex)




A lot of minimizing curves

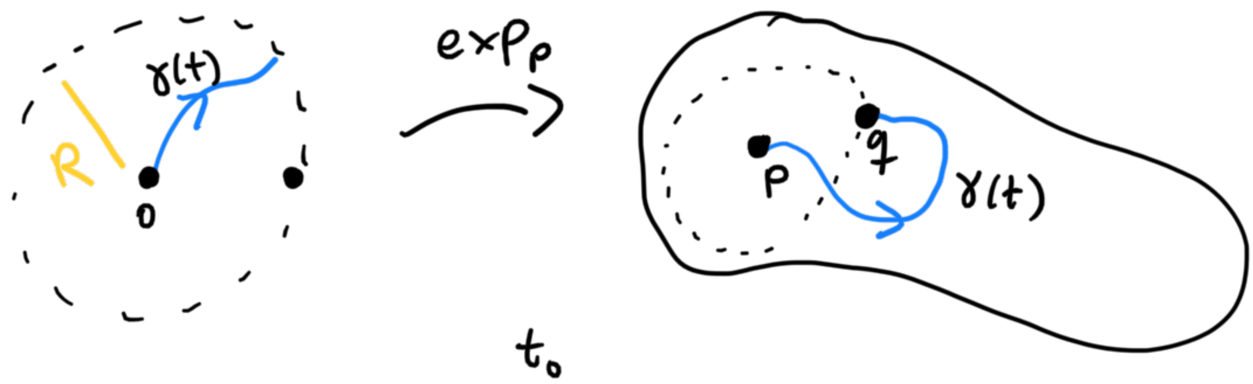
ex)


**Pf of Thm:** Take $\mathcal{U}$ s.t. $\exp_p : B_\varepsilon(0) \to \mathcal{U}$ is diffeo. Let $q \in \mathcal{U}$.

Let $\gamma(t)$ be arbitrary curve from $p$ to $q$.

$\gamma(0) = p$, $\gamma(t_0) = q$.

Let $R = r(q)$. Let $t_0$ be first time when $r(\gamma(t)) = R$.




$$L\left(\gamma|_{[0,t_0]}\right) = \int_0^{t_0} |\dot\gamma|_g \, dt$$

$$= \int_0^{t_0} |\nabla r|_g |\dot\gamma|_g \, dt \qquad |\nabla r| = 1$$

$$\geq \int_0^{t_0} \langle \nabla r, \dot\gamma \rangle_g \, dt \qquad \text{Cauchy-Schwarz}$$

$$= \int_0^{t_0} \left( g_{ik} g^{il} \partial_l r \, \dot\gamma^k \right) dt$$

$$= \int_0^{t_0} \partial_k r \, \dot\gamma^k \, dt$$

$$= \int_0^{t_0} \frac{d}{dt} r(\gamma(t)) \, dt \qquad \text{Chain rule}$$

$$= r(\gamma(t_0)) - r(\gamma(0)) \qquad \text{FTC}$$

$$= R.$$

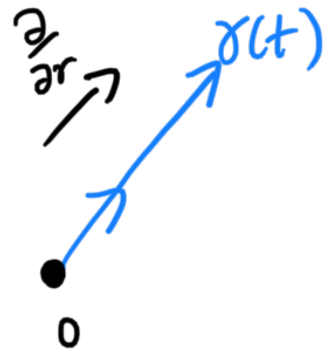
Equality in Cauchy Schwarz:

$$|u||v| = |\langle u, v\rangle| + \frac{1}{|v|^2} \Big| |v|^2 u - \langle u, v \rangle v \Big|^2.$$

Equality $\Leftrightarrow$ extra square term vanishes

$\Leftrightarrow$ $\dot{\gamma}$ parallel to $\nabla r$.

Normalize $\gamma(t)$ s.t. $\dot{\gamma} = \nabla r|_{\gamma(t)} = \frac{\partial}{\partial r}\Big|_{\gamma(t)}$.




$\gamma$ is straight line in normal coords, hence $\gamma(t)$ is a geodesic. $\square$

<u>Def</u>: $B_r(p) = \{q \in M : d(p,q) < r\}$ geodesic ball

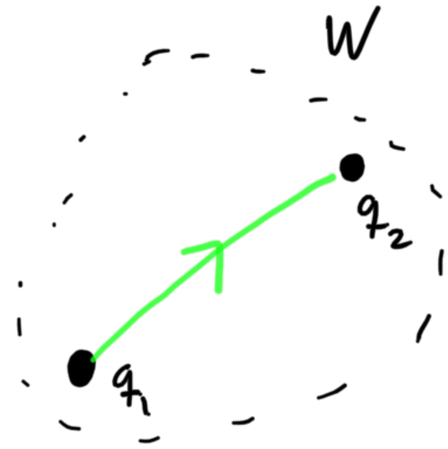
<u>Cor</u>: $B_r(p) = \exp_p(B_r^{g_{Euc}}(0))$ for small $r$.

<u>Prop</u>: $\forall p \in M$, $\exists \delta > 0$ and nbhd $W$ of $p$ which is a uniformly $\delta$-normal nbhd:

- $\forall q_1, q_2 \in W$, then $q_2 \in \exp_{q_1}(B_\delta(0))$.

$\Rightarrow q_2 = \exp_{q_1}(v)$, $v \in T_{q_1}M$ s.t. $|v|_g < \delta$.

- $\forall q \in W$ and $v \in T_qM$ with $|v|_g = 1$, the geodesic with $\gamma(0) = q$, $\dot{\gamma}(0) = v$ exists on $[0, \delta)$.




<u>Cor</u>: Let $p, q \in M$.

Let $\gamma$ be length minimizing curve from $p$ to $q$.

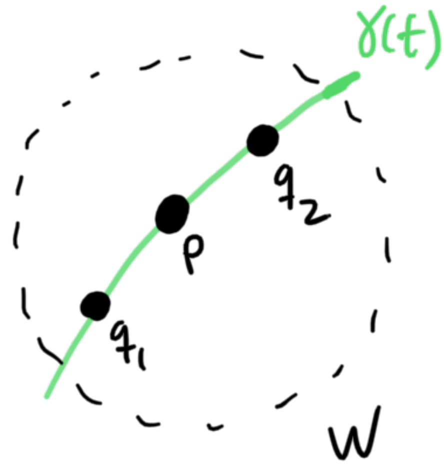
Then $\gamma$ is a geodesic.

($\Rightarrow$ length minimizers are smooth)

Pf: Let $p=\gamma(t_0)$, $W$ unif $\delta$-normal nbhd of $p$.

$q_1=\gamma(t_1)$, $q_2=\gamma(t_2)\in W$.

They are connected by length minimizing $\gamma$.




$\Rightarrow \gamma\big|_{[t_1,t_2]} = \exp_{q_1}(tv)$ some $v\in T_{q_1}M$

uniqueness in $W$

$\Rightarrow \gamma$ solves geo eqn in nbhd of $t_0$.

$t_0$ arbitrary $\Rightarrow \gamma$ solves geo eqn. $\square$

Pf of $\delta$-normal nbhd:

Let $p\in M$, $\mathcal{U}$ coord chart centered at $p$

$$E: \underset{\subseteq M}{\mathcal{U}} \times \underset{\subseteq \mathbb{R}^n}{V} \to M\times M$$

$$E(x,v) = (x, \exp_x(v)).$$

$$DE\big|_{(p,0)} = \begin{pmatrix} \frac{\partial x^i}{\partial x^j} & 0 \\ \frac{\partial \exp_x{}^i}{\partial x^j} & \frac{\partial \exp_x{}^i}{\partial v^j} \end{pmatrix} = \begin{pmatrix} I & 0 \\ * & I \end{pmatrix}.$$

invertible

linearized $\exp_p$ @ vector $0$

$$\text{IFT} \Rightarrow \exists\ \underset{\subseteq M}{W_0} \times \underset{\subseteq \mathbb{R}^n}{B_\varepsilon^{Euc}(0)} \xrightarrow[E \text{ diffeo}]{} \underset{\subseteq M\times M}{\mathcal{V}}$$

Let $q\in W_0$, then: $\exp_q(B_\varepsilon^{Euc}(0))$ well-defined normal nbhd.

($\exp_q$ defined on $B_\varepsilon$, + injective $E(q,v)=E(q,w)\Rightarrow v=w$)

Rewrite using $g_{ij}$ : $v \in B_\varepsilon^{Euc}(0) \longleftrightarrow v \in T_qM$

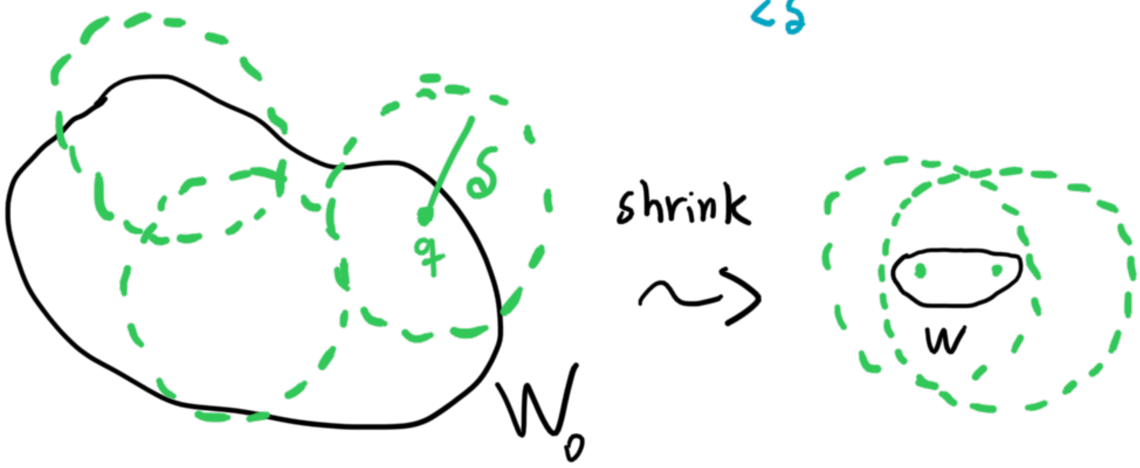
$$C^{-1}|v|_{g_{Euc}} \leq |v|_{g(q)} \leq C\,|v|_{g_{Euc}}$$

since $\overline{W} \subseteq (M,g)$ is compact, $\exists$ unif bounds on $g_{ij}(x)$.

$\delta = \frac{\varepsilon}{C}$ , $|v|_g < \delta \implies v \in B_\varepsilon^{Euc}(0)$.

$\implies \forall q \in W_0$, then $\exp_q(B_\delta^g(0))$ well-defn normal nbhd.

$\therefore$ $\underset{v \in T_qM}{|v|_g} < \delta \implies \exp_q(\underbrace{tv}_{<\delta})$ exist $0 \leq t \leq 1$.




$\implies W \subseteq \exp_q(B_\delta^g(0))$, for any $q \in W$.

$\implies \forall q \in W$, $u \in T_qM$ with $|u|_g = 1$, then

$\exp_q(\underbrace{t\,u}_{<\delta})$ exists $0 \leq t < \delta$.

$\square$