ex) Dual bundle

- $E \to M$ vector bundle with trans fun $(\mathcal{U} \cap \mathcal{V}, c_{\mathcal{UV}})$.
- $E^* \to M$ is vector bundle with trans fun $(\mathcal{U} \cap \mathcal{V}, (c_{\mathcal{UV}}^{-1})^T)$.

<u>Notation:</u>

- local sections of E denoted $s_{\mathcal{U}}^i(x)$ ← upper index
- local sections of E^* denoted $\phi_i^{\mathcal{U}}(x)$ ← lower index

<u>Transformation Law:</u>

- $s_{\mathcal{U}}^i = c_{\mathcal{UV}}{}^i{}_p \, s_{\mathcal{V}}^p$ $\qquad s_{\mathcal{U}} = \begin{pmatrix} s_{\mathcal{U}}^1 \\ \vdots \\ s_{\mathcal{U}}^k \end{pmatrix}$ ← column vector
- $\phi^{\mathcal{U}}_i = \phi^{\mathcal{V}}_p \, \underbrace{c_{\mathcal{VU}}}_{c_{\mathcal{UV}}^{-1}}{}^p{}_i$ $\qquad \phi^{\mathcal{U}} = (\phi^{\mathcal{U}}_1, \dots, \phi^{\mathcal{U}}_k)$ ← row vector

transpose acts on the right

$$\phi^{\mathcal{U}} = \phi^{\mathcal{V}} c_{\mathcal{VU}}$$ transformation law on E^*

If $s \in \Gamma(M, E)$, $\phi \in \Gamma(M, E^*)$

Then $\phi(s) := \phi_i s^i$ is well-defn scalar function.

Indeed:
$$s_{\mathcal{U}}^i \phi^{\mathcal{U}}_i = c_{\mathcal{UV}}{}^i{}_k \, s_{\mathcal{V}}^k \, \phi^{\mathcal{V}}_p \, c_{\mathcal{VU}}{}^p{}_i$$
$$= s_{\mathcal{V}}^k \phi^{\mathcal{V}}_k$$
well-defn indep of choice of local expression

Point of view of frames:

- $\{e_a\}$ local frame for $E \to M$
- $\{e^a\}$ denotes local frame for $E^* \to M$

$e^a|_p \in E|_p$ is dual vector to $e_a|_p \in E^*|_p$

i.e. $e^a(e_b) = \delta^a{}_b$.

Then $s \in \Gamma(M, E)$, $\phi \in \Gamma(M, E^*)$ are written

$$s \overset{loc}{=} s^i e_i, \quad \phi \overset{loc}{=} \phi_i e^i$$

and pairing is: $\phi(s) = (\phi_i e^i)(s^k e_k)$
$= \phi_i s^k e^i(e_k)$
$= \phi_i s^k \delta^i{}_k = \phi_i s^i$ ✓

ex) T^*M cotangent bundle

Sections called 1-form, $\alpha \in \Omega^1(M)$

$$\alpha = \alpha_i \, dx^i = \alpha^{\mathcal{U}}_i \, dx^i_{\mathcal{U}}$$

$$\tilde{\alpha}_i = \alpha_p \frac{\partial x^p}{\partial \tilde{x}^i}$$

on overlap $(\mathcal{U}, x^i)$, $(\tilde{\mathcal{U}}, \tilde{x}^i)$

$$d\tilde{x}^i = \frac{\partial \tilde{x}^i}{\partial x^p} dx^p$$

Pairing between VF $W = W^i \partial_i$ and 1-form $\alpha = \alpha_i dx^i$:

$\alpha(W) = \alpha_i W^i \in C^\infty(M, \mathbb{R})$ scalar function.

$\{dx^i\}_{i=1}^n$ is dual basis to $\left\{\frac{\partial}{\partial x^i}\right\}_{i=1}^n$:

$$dx^i\left(\frac{\partial}{\partial x^k}\right) = \delta^i{}_k.$$

ex) Given $f \in C^\infty(M, \mathbb{R})$, $df \in \Omega^1(M)$ given by

$$df \overset{loc}{=} \frac{\partial f}{\partial x^i} dx^i \qquad \frac{\partial f}{\partial \tilde{x}^i} = \frac{\partial x^p}{\partial \tilde{x}^i} \frac{\partial f}{\partial x^p}.$$

Tensor Product:

$E \to M$, $E' \to M$ vector bundles rank k, k'

$E \otimes E' \to M$ has fibers $(E \otimes E')|_p = E|_p \otimes E'|_p$

E has local frame $\{e_i\}_{i=1}^r$ (can take refinement $\mathcal{U}_\alpha \cap \mathcal{U}'_\alpha$ of trivs)

E' has local frame $\{e'_\alpha\}_{\alpha=1}^{r'}$

$E \otimes E'$ has local frame $\{ e_i \otimes e'_\alpha \}$

$E \otimes E'$ has rank rr'

In components: $s = s^{i\alpha} e_i \otimes e'_\alpha$,

$$s_{\mathcal{U}}^{i\alpha} = c_{\mathcal{U}V}{}^i{}_\kappa \, c'_{\mathcal{U}V}{}^\alpha{}_\beta \, s_V^{\kappa\beta} \text{ on } \mathcal{U} \cap V$$

Direct Sum:

$E \oplus E'$ has local frame $\{ e_i \oplus e'_\alpha \}$

$E \oplus E'$ has rank $r + r'$

Homomorphism:

$$\mathrm{Hom}(E, E') = E^* \otimes E'$$

call $s \in \Gamma(M, \mathrm{Hom}(E,E'))$ bundle homomorphism

$s \in \Gamma(M, \underset{\substack{\shortparallel \\ \mathrm{End}\, E}}{\mathrm{Hom}(E,E)})$ endomorphism of E

Def: bundle isomorphism is section $h \in \Gamma(M, \mathrm{Hom}(E,E'))$ s.t. $h = h^\alpha{}_i \, e^i \otimes e'_\alpha$ with $[h^\alpha{}_i]$ invertible matrix at each pt.

Def: E and E' are isomorphic if $\exists$ bundle isomorphism.

Pullback Bundle:

$E \to N$ bundle, $N = \bigcup \mathcal{U}_\alpha$ triv cover, $c_{\mathcal{U}V}$ trans fun

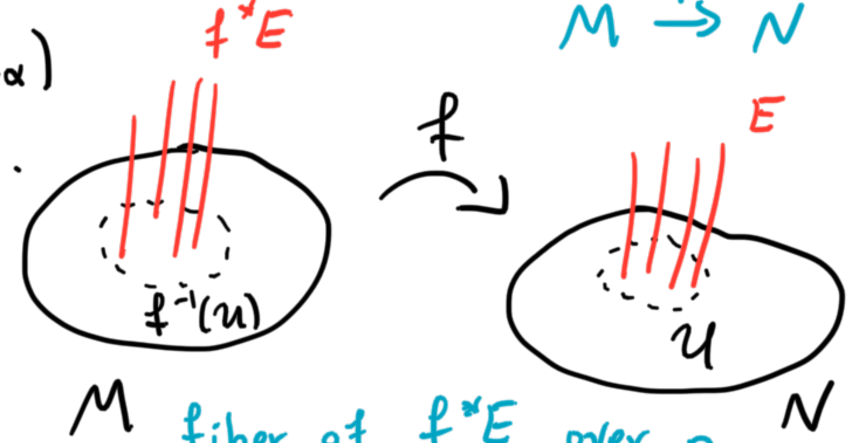
$f : M \to N$ map

Pullback bundle: $f^*E \to M$ given by

$$\begin{tikzcd} f^*E \arrow[d] & E \arrow[d] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

triv cover: $M = \bigcup f^{-1}(\mathcal{U}_\alpha)$

trans fun: $\tilde{c}_{\mathcal{U}V} = c_{\mathcal{U}V} \circ f$.



fiber of f^*E over p is fiber of E over $f(p)$.

ex) $\gamma : [0,1] \to M$

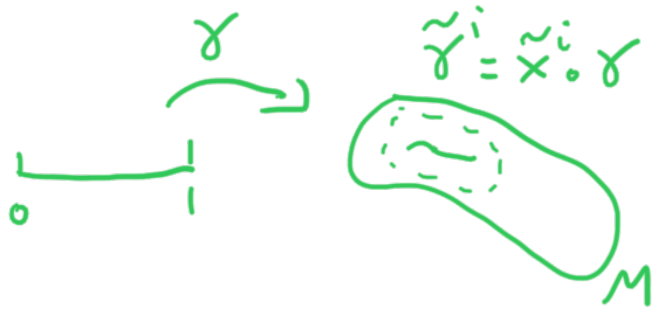
$\gamma^* TM \to [0,1]$

$\frac{d\gamma}{dt} \in \Gamma(\gamma^* TM)$

$(\mathcal{U}, x)$, $(\tilde{\mathcal{U}}, \tilde{x})$ coords on M

$\gamma \overset{loc}{=} (x^1(t), \dots, x^n(t))$

$\gamma \overset{loc}{=} (\tilde{x}^1(t), \dots, \tilde{x}^n(t))$



$$\frac{d\tilde{\gamma}^i}{dt} = \frac{d\tilde{x}^i}{dt} = \frac{\partial \tilde{x}^i}{\partial x^k}\frac{dx^k}{dt} \Rightarrow \frac{d\tilde{\gamma}^i}{dt} = \frac{\partial \tilde{x}^i}{\partial x^k}(\gamma(t))\frac{d\gamma^k}{dt}.$$

Correct transformation for $\gamma^* TM$

Orientability:

A bundle $E \to M$ is orientable if it admits trivializing cover s.t. trans fun c_{UV} all satisfy

$$\det c_{UV} > 0.$$

Main ex: TM tangent bundle.

If TM orientable, say M is orientable.

Suppose M orientable with coords $(\mathcal{U}, x^i)$ s.t. on overlaps $\mathcal{U} \cap \tilde{\mathcal{U}}$ then $\det\left[\frac{\partial \tilde{x}^i}{\partial x^k}\right] > 0$.

Let $\{e_1, \dots, e_n\}$ be an arbitrary basis of T_pM.

Write: $e_a = e^i{}_a \, \partial/\partial x^i$ (if $p \in \mathcal{U}$)

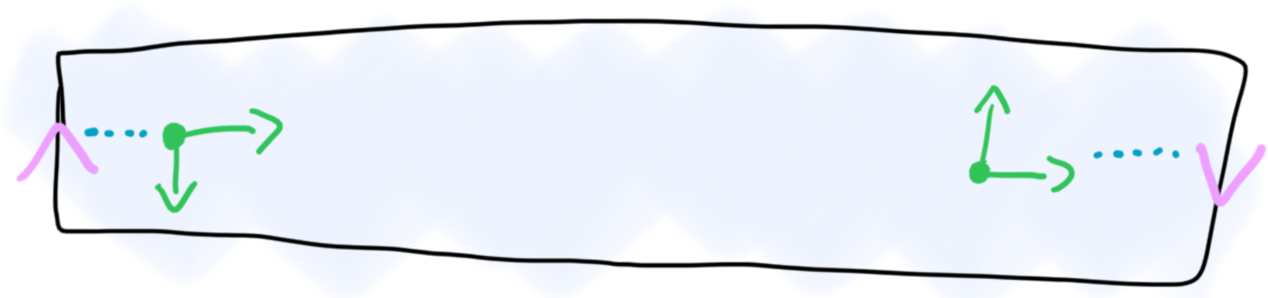
<u>Declare:</u> $[e_a]$ positively oriented if $\det[e^i{}_a] > 0$.

<u>well-defined:</u> if $p \in \mathcal{U} \cap \tilde{\mathcal{U}}$ and $\det[e^i{}_a] > 0$, then

$$e_a = e^i{}_a \frac{\partial}{\partial x^i} = \tilde{e}^i{}_a \frac{\partial}{\partial \tilde{x}^i}, \quad \tilde{e}^i{}_a = \frac{\partial \tilde{x}^i}{\partial x^p} e^p{}_a$$

$$\det[\tilde{e}^i{}_a] = \det\left[\frac{\partial \tilde{x}}{\partial x}\right]\det[e^p{}_a] > 0.$$

ex) Möbius strip not orientable



- If $\gamma : [0,1] \to M$ loop on oriented M,

  $\{ e_a(t) \}$ cont moving frame along $\gamma(t)$,

then: $[e_a(0)]$ pos oriented $\Rightarrow$ $[e_a(1)]$ pos oriented

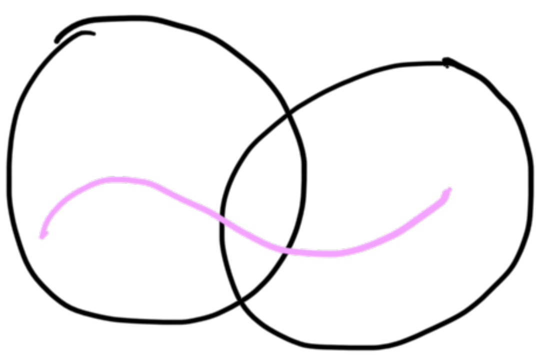
(not true for Möbius)

In chart: $\det [e_a(t)] \neq 0$ cannot pass through zero

on overlap:

$$\underbrace{\det [e_a(t)]}_{>0} = \underbrace{\det \left[\frac{\partial x}{\partial \tilde{x}} \right]}_{>0} \det [\tilde{e}_a(t)]$$

$[e_a(t)]$



$[\tilde{e}_a(t)]$