MATH 309

Solutions – Convexity

1. The bounding line is determined by the points $(1/4, 0)$ and $(0, 1)$; $H^+((4, 1), 1)$ is the half-plane above this line, and $H^-((4, 1), 1)$ is the half-plane below it.

2. The bounding line again is determined by the points $(1/4, 0)$ and $(0, 1)$; but this time, $H^+((4, 1), 1)$ is the half-plane *below* this line, and $H^-((4, 1), 1)$ is the half-plane above it.

3. We may assume that for some $\mathbf{v} \in \mathbb{R}^n$, $\mathbf{v} \neq 0$, and $c \in \mathbb{R}$, the half-space is

$$H = H^+(\mathbf{v}, c) = \{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{x}, \mathbf{v} \rangle \geqslant c\}.$$

We have to show that for any $\mathbf{x}, \mathbf{y} \in H$ and for any $\lambda \in [0, 1]$, $\lambda\mathbf{x} + (1 - \lambda)\mathbf{y} \in H$. Using the linearity of the dot product and that both λ and $(1 - \lambda)$ are non-negative, this can be seen by

$$\begin{aligned} \langle \lambda\mathbf{x} + (1 - \lambda)\mathbf{y}, \mathbf{v} \rangle &= \lambda\langle \mathbf{x}, \mathbf{v} \rangle + (1 - \lambda)\langle \mathbf{y}, \mathbf{v} \rangle \\ &\geqslant \lambda c + (1 - \lambda)c = c. \end{aligned}$$

4. Let $S = (-\infty, 0] \cup [2, \infty)$, and take $x = 1$. Then there are two points in S which are closest to x: 0 and 2. Another example is $S = (0, \infty)$, and $x = -1$, in which case there is no closest point to x in S.

5. With the standard notation, S is the intersection of four half-planes, which are

$$H^+((0, 1), 0)\ ,\ H^-((0, 1), 1)\ ,\ H^+((1, 0), 0)\ ,\ H^-((1, 0), 1).$$

6. a) The vertices of the convex hull are $(0, 0), (2, 2), (0, 3)$, thus, the convex hull is a triangle. The remaining two points can be written as

$$\begin{aligned} (1, 1) &= \frac{1}{2}\,(0, 0) + \frac{1}{2}\,(2, 2), \\ (0, 2) &= \frac{1}{3}\,(0, 0) + \frac{2}{3}\,(0, 3). \end{aligned}$$

b) The vertices of the convex hull are $(1, 0), (-1, 0), (0, 1), (0, -1)$, thus $\mathrm{conv}(Y)$ is a square rotated by 45 degrees. The remaining point can be expressed for example as

$$(0, 0) = \frac{1}{2}\,(-1, 0) + \frac{1}{2}\,(1, 0).$$

7. The set X is the union of two infinite lines, whose slopes are 1 and -1, intersecting at the origin. The convex hull, $\mathrm{conv}(X)$ is the whole plane. To formally see this, let (x_1, x_2) be an arbitrary point in $\mathbb{R}^2$, and assume that $x_1 \geqslant x_2$. Then the vertical segment with endpoints (x_1, x_1) and $(x_1, -x_1)$ has its endpoints in X, and the point (x_1, x_2) lies on it.

8. Probably the simplest example is the three sides of a non-degenerate triangle (i.e. three segments). Any two of them intersects in a point (a vertex of the triangle), but the intersection of all three is clearly empty.

9.* The set of all rational numbers $\mathbb{Q}$ is such an example. Another example (and, in some sense, the minimal example) is the set of dyadic rationals, i.e. the set

$$\left\{\frac{n}{2^m}\right\},\ n \in \mathbb{Z},\ m \in \mathbb{N}.$$

Note that the property missing for convexity is closedness; thus, if we assume that a set is *closed* and has the specified property about halving points, then it is also convex.