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ON THE NUMBER OF SUMS AND PRODUCTS

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Abstract

A new lower bound on $\max\{|A+A|, |A\cdot A|\}$ is given, where A is a finite set of complex numbers.

1. *Introduction*

Let A be a finite subset of complex numbers. The *sum-set* of A is $A+A=\{a+b : a,b\in A\}$, and the *product-set* is $A\cdot A=\{a\cdot b : a,b\in A\}$. Erdős and Szemerédi [**7**] proved the inequality

$$\max(|A+A|, |A\cdot A|) \geqslant c|A|^{1+\varepsilon}$$

for a small but positive ε, where A is a subset of integers. They conjectured that

$$\max(|A+A|, |A\cdot A|) \geqslant c|A|^{2-\delta}$$

for any positive δ. (In this paper, c stands for the general constant. Some authors use the $n \ll m$ or $n \gg m$ notation instead of our $n \leqslant cm$ or $n \geqslant cm$.)

After improvements given in [**9**], [**8**], and [**3**], the best bound so far has been obtained by Elekes [**4**], who showed that $\varepsilon \geqslant 1/4$ if A is a set of real numbers. His result was extended to complex numbers in [**13**] and [**11**]. For further results and related problems, we refer the reader to [**1**] and [**5**].

In this paper, we prove the following theorem.

Theorem 1. *There is a positive absolute constant c such that, for every n-element set A,*

$$\frac{cn^{14}}{\log^3 n} \leqslant |A+A|^8 \cdot |A\cdot A|^3,$$

whence $cn^{14/11}/\log^{3/11} n \leqslant \max\{|A+A|, |A\cdot A|\}$.

Nathanson and Tenenbaum [**10**] proved that the product set should be large, namely $|A|^{2-\varepsilon}$, if the sumset is at most $3|A|-4$. Chang [**2**], and independently Elekes and Ruzsa [**6**], proved a similar bound if the sumset is at most $c|A|$. As a consequence of Theorem 1, we obtain the following corollary.

Corollary 1. *If $|A|=n$ and $|A+A| \leqslant Cn$, then $|A\cdot A| \geqslant cn^2/\log n$.*

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2. *Proof*

Our proof is based on the following estimates of the number of incidences between lines and points.

THEOREM 2 (Szemerédi and Trotter [**12**]). *The maximum number of incidences between n points and m straight lines of the real plane is $O(n^{2/3}m^{2/3} + n + m)$.*

COROLLARY 2 (Szemerédi and Trotter [**12**]). *Given a set of n points on the real plane, the number of k-rich lines (that is, lines incident to at least k points) is $O(n^2/k^3 + n/k)$.*

In the proof of Theorem 1 we use Theorem 2 and Corollary 2 on Cartesian products only; similar statements are easy to prove for complex lines in the complex plane. (The general case has recently been solved by Tóth [**13**].) The following lemma has been proved but not published by the author.

LEMMA 1. *Given two sets of complex numbers S_1 and S_2 with sizes $|S_1| = n_1$ and $|S_2| = n_2$, let $S = S_1 \times S_2$ be the Cartesian product. The maximum number of incidences between S and m complex lines of the complex plane is $O((n_1 n_2)^{2/3} m^{2/3} + n_1 n_2 + m)$.*

Proof of Theorem 1. If $|A \cdot A| = t$, then the number of pairs $(a_i, a_j), (a_u, a_v)$ such that $a_i \cdot a_j = a_u \cdot a_v$ (where $a_i, a_j, a_u, a_v \in A$) is at least cn^4/t. Then the number of pairs $(a_i, a_v), (a_u, a_j) \in A \times A$, where $a_i/a_v = a_u/a_j$, is at least cn^4/t as well. Let us partition the elements of $A \times A$ into classes (lines) $L_1, L_2, \ldots, L_k$ using the relation $(a_i, a_j) \sim (a_u, a_v)$ if and only if $a_i/a_j = a_u/a_v$. Each class is a collection of collinear points, and the line through them contains the origin $(0, 0)$. If l_i denotes the size of L_i, then

$$\sum_{i=1}^{k} \binom{l_i}{2} \geqslant \frac{cn^4}{t}.$$

We partition these lines into sets $C_1, C_2, \ldots, C_s$ ($s \leqslant \log n^2$) with respect to their 'squared' sizes. Then $L_i \in C_j \iff 2^{2(j-1)} < \binom{l_i}{2} \leqslant 2^{2j}$. There are at most $\log n^2$ sets, so there is at least one set, C_j, which covers many elements. Let X_j be a set of all pairs $((a_\nu, a_\mu), (a_\varrho, a_\rho))$ such that there exists L_i in C_j with (a_ν, a_μ) and (a_ϱ, a_ρ) both in L_i. Then at least one of the sets X_j is large. Also,

$$|X_j| = |\{(a_\nu, a_\mu), (a_\varrho, a_\rho) : (a_\nu, a_\mu) \in L_i, (a_\varrho, a_\rho) \in L_i, L_i \in C_j\}| \geqslant \frac{cn^4}{t \log n},$$

and therefore

$$2^{2j}|C_j| \geqslant \frac{cn^4}{t \log n}. \tag{2.1}$$

This is the key element of the proof: every point of $A \times A$ is incident to at least $|C_j|$ lines, each of them incident to at least 2^{j-1} points of $(A+A) \times (A+A)$. Indeed, the translated lines $(a_u, a_v) + L$ with L in C_j are incident to (a_u, a_v), and the points of the lines are points from $(A + A) \times (A + A)$ (see Figure 1). We denote the set of translated lines by $\mathcal{L}$, as follows:

$$\mathcal{L} = \{(a_u, a_v) + L : L \in C_j, (a_u, a_v) \in A \times A\}.$$

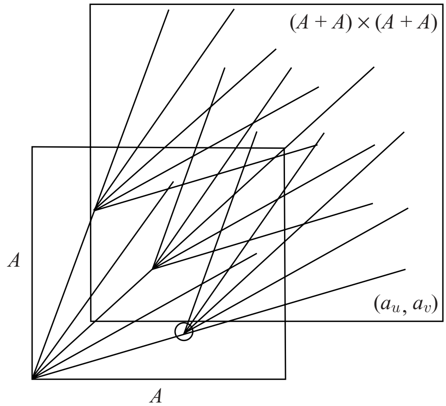

FIGURE 1. *Translates of the lines of* C_j.

Because of Corollary 2, the number of 2^{j-1}-rich lines (that is, lines incident to at least 2^{j-1} points) on $(A+A)\times(A+A)$ is

$$O\left(\frac{|A+A|^4}{(2^{j-1})^3}+\frac{|A+A|}{(2^{j-1})}\right).$$

The first term is always larger than the second because $|A+A|>|A|$ and $2^{j-1}\leqslant|A|$. Therefore,

$$|\mathcal{L}|\leqslant\frac{c|A+A|^4}{(2^{j-1})^3}.$$

Applying the bound from Theorem 2 to the number of incidences I between $\mathcal{L}$ and the n^2 points of $A\times A$, we have

$$I=O\big(|\mathcal{L}|^{2/3}(n^2)^{2/3}+|\mathcal{L}|+n^2\big).$$

Therefore,

$$n^2|C_j|\leqslant c|\mathcal{L}|^{2/3}n^{4/3}, \tag{2.2}$$

or

$$n^2|C_j|\leqslant c|\mathcal{L}|, \tag{2.3}$$

or

$$n^2|C_j|\leqslant cn^2. \tag{2.4}$$

The right-hand side of (2.2) is always at least cn^2, and therefore (2.2) includes case (2.4). The next step is to see that (2.2) covers case (2.3) as well. Let us suppose that, on the contrary,

$$|\mathcal{L}|^{2/3}n^{4/3}<|\mathcal{L}|.$$

Then

$$n^{4/3}<|\mathcal{L}|^{1/3}\to n^4<|\mathcal{L}|,$$

but this is not possible, since $\mathcal{L}$ consists of n^2 translates of less than n^2 lines.

Now we are ready for the final step of the proof. It follows from (2.1), that

$$2^{2j-2} \geqslant \frac{cn^4}{t \log n |C_j|}. \tag{2.5}$$

Putting (2.2) and (2.5) together, we have

$$\begin{aligned} n^2|C_j| &\leqslant c|\mathcal{L}|^{2/3} n^{4/3} \leqslant c\left(\frac{|A+A|^4}{(2^{j-1})^3}\right)^{2/3} n^{4/3} = c\frac{|A+A|^{8/3}}{2^{2j-2}} n^{4/3} \\ &\leqslant c\frac{|A+A|^{8/3}}{(n^4/t\log n|C_j|)} n^{4/3}, \end{aligned}$$

which gives

$$\frac{cn^{14}}{\log^3 n} \leqslant |A+A|^8 \cdot t^3,$$

as stated. □

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