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A Note on a Question of Erdős and Graham

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We give a quantitative proof that, for sufficiently large N, every subset of $[N]^2$ of size at least δN^2 contains a square, *i.e.*, four points with coordinates $\{(a,b),(a+d,b),(a,b+d),(a+d,b+d)\}$.

1. Introduction

In this note we prove a generalization of Szemerédi's theorem about arithmetic progressions of length four [12]. This generalization, Theorem 1.1, was first considered by Ron Graham in 1970 and conjectured by him and Erdős (published in [2] and [1]). Using Szemerédi's deep theorem [11] about arithmetic progressions of length k, Ajtai and Szemerédi [1] proved a simpler statement: for sufficiently large N, every subset of $[N]^2$ of size at least δN^2 contains three points with coordinates $\{(a,b),(a+d,b),(a,b+d)\}$. ($[N]=\{0,1,2,\ldots,N-1\}$). Later Fürstenberg and Katznelson proved a much stronger general theorem [3] (see Theorem 3.1), but their proof does not give an explicit bound as it uses ergodic theory. After giving an analytic proof for Szemerédi's theorem, Tim Gowers again raised the question of finding a quantitative proof for Graham's question [5, 6]. Using a recent result of Frankl and Rödl we give a combinatorial proof for this theorem.

Theorem 1.1. *For any real number $\delta > 0$ there is a natural number $N_0 = N_0(\delta)$ such that for $N > N_0$ every subset of $[N]^2$ of size at least δN^2 contains a square, i.e., a quadruple of the form $\{(a,b),(a+d,b),(a,b+d),(a+d,b+d)\}$ for some integer $d \neq 0$.*

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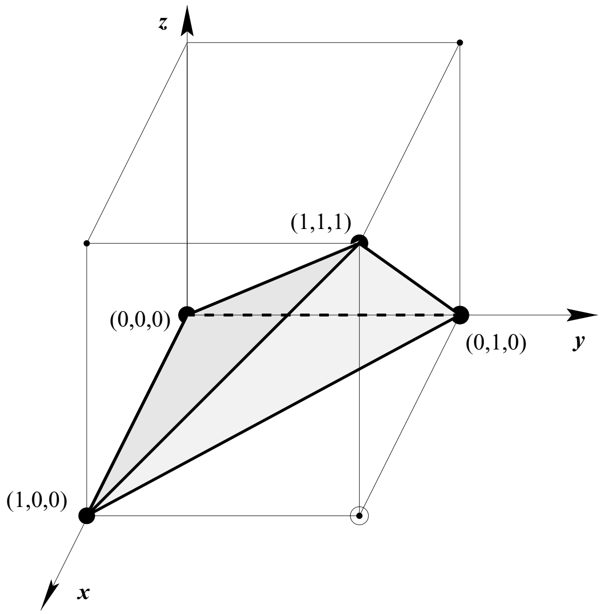

Figure 1. A quadruple of the form (1.1)

Before Theorem 1.1 we prove the following theorem.

Theorem 1.2. *For any real number $\delta > 0$ there is a natural number $N_0 = N_0(\delta)$ such that, for $N > N_0$, every subset of $[N]^3$ of size at least δN^3 contains a quadruple of the form*

$$\{(a,b,c),(a+d,b,c),(a,b+d,c),(a+d,b+d,c+d)\} \tag{1.1}$$

for some integer $d \neq 0$.

Proposition 1.3. *Theorem* 1.2 *implies Theorem* 1.1. □

Proof. Let us suppose that Theorem 1.1 is false. Then there is a real number $\delta > 0$ and, for every N, a subset S_N of $[N]^2$, such that $|S_N| > \delta N^2$ and S_N does not contain any square. For every S_N we can define a subset of $[N]^3$ by lifting up all the points of S_N into 3D:

$$S_N^* = \{(a,b,c) : (a,b) \in S_N, c \in [N]\}.$$

The size of S_N^* is larger than δN^3 and does not contain any quadruple of the form (1.1). This contradicts Theorem 1.2. □

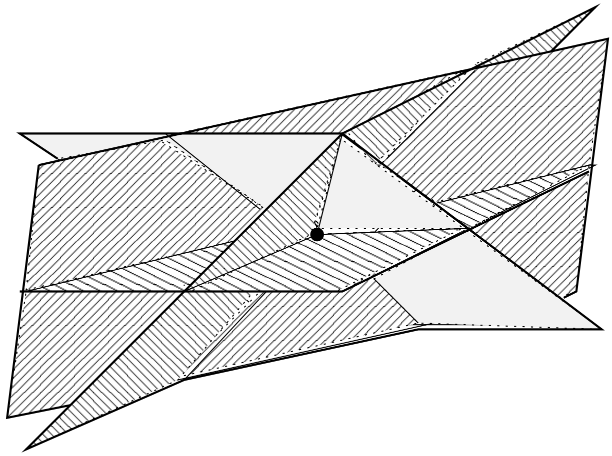

Figure 2. Every point of S defines $\binom{4}{3}$ edges in $\mathscr{H}$

2. Proof

Proof of Theorem 1.2. We define a three-uniform hypergraph $\mathscr{H}$. The vertex set $V(\mathscr{H})$ is a collection of planes:

$$\begin{aligned} a_i &= \{z = i\} && \text{and } V_1 = \{a_i : 0 \leqslant i \leqslant N-1\}, \\ b_i &= \{-x + z = i\} && \text{and } V_2 = \{b_i : -N+1 \leqslant i \leqslant N-1\}, \\ c_i &= \{-y + z = i\} && \text{and } V_3 = \{c_i : -N+1 \leqslant i \leqslant N-1\}, \\ d_i &= \{x + y - z = i\} && \text{and } V_4 = \{d_i : -N+1 \leqslant i \leqslant 2N-2\}, \\ V(\mathscr{H}) &= V_1 \cup V_2 \cup V_3 \cup V_4. \end{aligned}$$

These are the planes parallel with the faces of any simplex given by (1.1) and have points from $[N]^3$. The edge set $E(\mathscr{H})$ is defined by a point set $S \subset [N]^3$. Three distinct vertices v_1, v_2, and v_3 form an edge if the intersection point of the corresponding planes p_1, p_2 and p_3 is in S, that is,

$$E(\mathscr{H}) = \{(v_1, v_2, v_3) : v_i \in V (1 \leqslant i \leqslant 3), p_1 \cap p_2 \cap p_3 \in S\}.$$

$\mathscr{H}$ is a 4-partite hypergraph with classes V_1, V_2, V_3, and V_4. We are going to show that if S does not contain any quadruple like (1.1), then $|E(\mathscr{H})|$ – and therefore also $|S|$ – is $o(N^3)$. This will prove Theorem 1.2.

The next conjecture is a special case of a more general conjecture of Frankl and Rödl [8]. A subgraph in a k-uniform hypergraph is a *complete subgraph* if it has at least $k+1$ vertices and all k-tuples of its vertices are edges.

Conjecture 2.1. *Given an integer $k \geqslant 2$. If $\mathscr{G}$ is a k-uniform hypergraph such that every edge is an edge of exactly one complete subgraph, then the number of edges $|E(\mathscr{G})|$ is $o(|V(\mathscr{G})|^k)$.*

For $k = 2$ the conjecture is equivalent to the so-called (6,3)-theorem proved by Ruzsa and Szemerédi [10], and the $k = 3$ case was proved by Frankl and Rödl [8].

Theorem 2.2. (Frankl and Rödl) *If $\mathscr{G}$ is a 3-uniform hypergraph such that every edge is an edge of exactly one complete subgraph, then the number of edges $|E(\mathscr{G})|$ is $o(|V(\mathscr{G})|^3)$.*

Remark. In their proof Frankl and Rödl applied Szemerédi's Regularity Lemma; therefore here we cannot achieve more than a tower-type upper bound on N_0 in Theorem 1.1. (For the details of why, in general, the Regularity Lemma gives only a weak bound, we refer to the paper of Gowers [4].)

In $\mathscr{H}$ four vertices a_i, b_j, c_k, and d_l form a complete subgraph if any triple has its intersection point in S. If the planes are not concurrent planes, *i.e.*, $a_i \cap b_j \cap c_k \cap d_l = \emptyset$, then a_i, b_j, c_k, and d_l is a quadruple like (1.1), *i.e.*, the intersection points of the triples form a simplex similar to {(0,0,0),(1,0,0),(0,1,0),(1,1,1)}, because the corresponding faces are parallel. Let us suppose that there is no such quadruple in S. Then every edge of $\mathscr{H}$ is an edge of exactly one complete subgraph, and $|E(\mathscr{H})| = o(N^3)$ by Theorem 2.2. □

3. Conjectures

If Conjecture 2.1 was true, then it would imply the following 'multidimensional Szemerédi theorem' [3].

Theorem 3.1. (Fürstenberg and Katznelson) *For any real number $\delta > 0$ and positive integers K, d there is a natural number $N_0 = N_0(\delta, K, d)$ such that for $N > N_0$ every subset of $[N]^d$ of size at least δN^d contains a homothetic copy of $[K]^d$.*

We state a special case of Conjecture 2.1. It would also imply Theorem 3.1 following the steps of the proof of Theorem 1.1 in higher dimensions, and as a plus there is some geometry which could be useful for a possible proof.

Conjecture 3.2. *For any real number $\delta > 0$ and positive integer d there is a natural number $N_0 = N_0(\delta, d)$ such that, for $N > N_0$, any set of N hyperplanes S and at least δN^d points, where every point is an element of at least $d+1$ hyperplanes, contains a simplex (i.e., $d+1$ distinct points such that any d-tuples are contained by a hyperplane from S).*

We close this note with a nice conjecture of Graham [7] which, if true, would give a sufficient condition for the existence of a square in an infinite lattice set.

Conjecture 3.3. (Graham) *Given a set of lattice points in the plane*

$$S = \{p_1, p_2, \ldots, p_i, p_{i+1}, \ldots\},$$

let us denote the distance of p_i from the origin by d_i. If

$$\sum_{i=1}^{\infty} \frac{1}{d_i^2} = \infty,$$

then S contains the four vertices of an axes-parallel square.

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References

[1] Ajtai, M. and Szemerédi, E. (1974) Sets of lattice points that form no squares. *Studia Scientiarium Mathematicarum Hungarica* **9** 9–11.
[2] Erdős, P. (1973) Problems and results on combinatorial number theory. In *A Survey of Combinatorial Theory* (*Proc. Internat. Sympos., Colorado State Univ., Fort Collins, Colo., 1971*), North-Holland, Amsterdam, pp. 117–138.
[3] Fürstenberg, H. and Katznelson, Y. (1991) A density version of the Hales–Jewett theorem. *J. d'Analyse Math.* **57** 64–119.
[4] Gowers, W. T. (1997) Lower bounds of tower type for Szemeredi's uniformity lemma. *GAFA* (*Geometric and Functional Analysis*) **7** 322–337.
[5] Gowers, W. T. (2001) A new proof of Szemerédi's theorem. *GAFA* (*Geometric and Functional Analysis*) **11** 465–588.
[6] Gowers, W. T. (1999) Rough structure and classification. *GAFA* (*Geometric and Functional Analysis*), *Special Volume: GAFA2000 'Visions in Mathematics', Tel Aviv*, Part I, 79–117.
[7] Graham, R. (1997) Conjecture 8.4.6. In *Discrete and Computational Geometry* (J. E. Goodman and J. O'Rourke, eds), CRC Press, Boca Raton, NY, p. 11.
[8] Frankl, P. and Rödl, V. (2002) Extremal problems on set systems. *Random Struct. Alg.* **20** 131–164.
[9] Roth, K. F. (1953) On certain sets of integers. *J. London Math. Soc.* **28** 245–252.
[10] Ruzsa, I. Z. and Szemerédi, E. (1976) Triple systems with no six points carrying three triangles. In *Combinatorics, Keszthely* (*Hungary*), Vol. 18 of *Colloquia Mathematica Societatis János Bolyai*, pp. 939–945.
[11] Szemerédi, E. (1975) On sets of integers containing no k elements in arithmetic progression. *Acta Arithmetica* **27** 199–245.
[12] Szemerédi, E. (1969) On sets of integers containing no four elements in arithmetic progression. *Acta Math. Acad. Sci. Hungar.* **20** 89–104. Also in *Number Theory* (*Colloq., János Bolyai Math. Soc., Debrecen, 1968*), North-Holland, Amsterdam, pp. 197–204.